Gravity, Self Duality, BF theory

Introduction: basic tools from differential geometry
Lecture notes, Paris, 2ol3

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Lecture notes - **Current update**: o8.2ol4, **Id-code** [BFGrav][VyD] - seminar in the context of the ERC Philosophy of Canonical Quantum Gravity, autumn 2ol3, Paris 7 University.[a] This is not the final version of the notes, mistakes may still appear and better updates will come later on. Any comments are welcome.

[a]The research leading to this lecture received funding from the European Research Council under the European Communitys Seventh Framework Programme (FP7/2007- 2013 Grant Agreement nu 263523).

Abstract In recent decades, a plethora of actions and theories (Einstein-Hilbert Palatini, Einstein-Cartan, Plebanski, MacDowell-Mansouri, BF theories, Chern -Simons...) have emerged as possible formulations and variations of the gravitational theory. Entanglement with topological terms (Holst, Euler, Pontryagin, Nieh-Yan), the notions of Hodge duality and self-duality are cornerstones of these constructions. The objective of this course is to make a taxonomy of the different useful actions for the theory of gravity as well as their associated mathematical objects. In the first part of the lecture we will focus on the following tools: Hodge duality, Self duality, Self dual formulation of Gravity, Ashtekar Gravity, Plebanski Gravity and finally Gravity as a topological BF theory. In particular, we will explore the tension between topological BF theories and the additional specific constraints (*simplicity constraints*) on them that allow us to recover a gravitational theory (with gravitational local degrees of freedom)

In this lecture, we explore the geometrical objects that appear for the relation between BF Topological Field theory and Gravity. Beyond the scope of this very brief introduction, is found three main directions and relationships that are respectively the relation of gravity with self-duality, topological theories and Cartan Geometry. For example, the 3D theory is expressed as a topological Chern-Simons theory whereas in 4D theory, the Einstein-Cartan theory can be pictured as a BF theory with additional constraints (simplicity constraints). Self-duality is very important, notice that original Plebanski and Ashtekar (anti)-self dual connections. More advanced topics such as the road to Penrose's twistors, conformal Einstein solutions or gravitational instantons are also based on the self-duality notion. The right geometrical landscape for gravity is described by Cartan Geometry. In the Einstein-Cartan action $\mathcal{L}^{\mathsf{4D}}_{\mathsf{EC}}[e,\omega] = \varepsilon_{IJKL} e^I \wedge e^J \wedge \mathsf{F}^{KL}$ this corresponds to give the interpretation of (e,ω) as the component of a Cartan connection. This open the way to symmetric models, the Poincaré, de Sitter and anti-de Sitter spacetimes and much Hybrid Intersections: MacDowell-Mansouri action describe in the context of Cartan Geometry, Plebanski formulation as a BF type theory, BF-formulation of MacDowell-Mansouri action. In the first part of these lecture, we leave apart the hybrids intersection, and also the connection with Cartan geometry, see Sharpe [43], Egeileh [18] or Wise [55] [56] [57]. We simply concentrate in this lecture to describe the basic tools, from differential geometry and Lie algebra perspective, encounter at the intersection bewteen Gravity, BF theory and Self duality.

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Hodge duality, Self-Duality. Self-duality is an important aspect which appears in various formulations of Yang-Mills theory and gravity. In the latter case, we describe this notion for Riemannian metrics. The (anti)self-dual Riemannian metrics are those whose Riemann tensor is (anti-)self-dual. The Hodge operator can act on internal or external space (if the tensor Riemannian curvature is written $\mathbf{R} = 1/2\mathbf{R}^{IJ}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes \mathbf{\Delta}_{IJ}$. where $\mathbf{\Delta}_{IJ}$ are the generators of the $\mathfrak{so}(1,3)$ Lie algebra. However if we do not introduce Lorentz indices here, we can think of two type of hodge duality whether the process is applied on the first pair or the second pair of indices in the expression of $\mathbf{R}_{\mu\nu\rho\sigma}$. The equivalent vocabulary, in order to notice this difference, is that we call the external hodge operator, the left handed sector $\star\mathbf{R}_{\mu\nu\rho\sigma} = (1/2)\varepsilon_{\mu\nu}{}^{\lambda\kappa}\mathbf{R}_{\lambda\kappa\rho\sigma} = +\mathbf{R}_{\mu\nu\rho\sigma}$ and $\star\mathbf{R}_{\mu\nu\rho\sigma} = (1/2)\varepsilon_{\mu\nu}{}^{\lambda\kappa}\mathbf{R}_{\lambda\kappa\rho\sigma} = -\mathbf{R}_{\mu\nu\rho\sigma}$. So that we denote $\star\mathbf{R}^{\blacktriangleleft} = \mathbf{R}^{\blacktriangleleft}$ if the Riemann curvature tensor is left-handed (self-dual).[2] We obtain the same condition on the Weyl tensor, and $\mathbf{R}_{\mu\nu} = 0$. Theses metrics are solutions of the Einstein equation.[3] The closely related topic of conformally (anti-)self-dual solutions is of great interest. Conformally (anti)-self-dual metrics are those whose whose Weyl tensor $\mathbf{C}_{\mu\nu\rho\sigma}$ is (anti)-self-dual.[4] The study of $\mathbf{C}_{\mu\nu\rho\sigma}$ gives directly the pure gravitational degree of freedom. Solution of conformally (anti)-self dual solution have been studied in particular by Penrose, Newman and Plebanski. These lines of thought are closely connected to deep features of Einstein gravity, in connection with the geometry of 4D Riemannian (Lorentzian) manifolds. In such a landscape, we notice the Petrov's classication of spaces defining gravitational fields, uses self-dual (and anti-self-dual) bivectors (two forms) as central objects. The solutions of the vacuum Einstein equations are classified according to algebraic properties of complex 3D matrix, see [35]. On the other hand, the famous Atiyah-Hitchin-Singer theorem is deeply connected to the topic. A metric $g_{\mu\nu}$ is Einstein ($g_{\mu\nu} \propto \mathbf{R}_{\mu\nu}$)[6] if and only if the restriction of the Levi-Civita connection $\Gamma^\rho_{\mu\nu}[g_{\mu\nu}]$ to the bundle of self-dual two-forms is self-dual. On the other side, self-dual formulation and complex representations are the ingredients found within the original Plebanski and Ashtekar gravity. The imortance of various complex representation of the Lorentz group $\mathsf{SO}(1,3)$ such as $\mathsf{SO}(3,\mathbb{C})$ or $\mathsf{SL}(2,\mathbb{C})$ (spinorial representation) is present in the work of Plebanski and Ashtekar. Therefore, complex representation seems to allow us to understand the real geometry of Einstein spaces. The first example of that development is the one of Plebanski gravity [78] see [36]. Einsteins GR is given as a formulation in which the dynamical object is a triple of self-dual two-forms. Basic objects of Plebanski's formulation of GR are self-dual two-forms. The original Plebanski formulation use spinor notations, *i.e* work with the group $\mathsf{SL}(2,\mathbb{C})$. The second formulation of interest is the Ashtekar gravity - [86,91], see [3] [4] [5]: new variables and Ashtekar's Hamiltonian formulation is in fact (see [22]) the phase space version of Plebansk gravity. The original Ashtekar formulation use spinor notations, *i.e* work with the group $\mathsf{SL}(2,\mathbb{C})$

Topological terms. We find different topological terms, the Euler invariant $\mathcal{L}_{\mathsf{Euler}}[\omega]$, the Pontrjagin invariant $\mathcal{L}_{\mathsf{Pontrjagin}}[\omega]$ and the Nieh-Yan invariant, $\mathcal{L}_{\mathsf{Nieh\text{-}Yan}}[e,\omega]$. The general action for first order Palatini gravity is described by a set of finite terms composed of the Holst term, the topological terms and finally the cosmological constant term - see various works [15] [27] [28] [29] which is compatible with the diffeomorphism invariance and the Lorentz invariance. $\mathcal{L}[e,\omega]$ is a sum of the following terms: $\mathcal{L}_{\mathsf{Einstein\text{-}Cartan}}[e,\omega]$, $\mathcal{L}_{\mathsf{Holst}}[e,\omega]$, $\mathcal{L}_{\mathsf{Euler}}[\omega]$, $\mathcal{L}_{\mathsf{Pontrjagin}}[\omega]$, $\mathcal{L}_{\mathsf{Nieh\text{-}Yan}}[e,\omega]$,

[2] and $\star\mathbf{R}^{\blacktriangleright} = -\mathbf{R}^{\blacktriangleright}$ if the Riemann curvature tensor is right-handed (anti-self-dual).

[3] For Lorentzian metrics: (Anti-)self-dual Lorentzian metrics are those whose Riemann tensor is (anti-)self-dual: $\star\mathbf{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\lambda\kappa}\mathbf{R}_{\lambda\kappa\rho\sigma} = i\mathbf{R}_{\mu\nu\rho\sigma}$ and $\star\mathbf{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\lambda\kappa}\mathbf{R}_{\lambda\kappa\rho\sigma} = -i\mathbf{R}_{\mu\nu\rho\sigma}$, so that $\star\mathbf{R}^{\blacktriangleleft} = i\mathbf{R}^{\blacktriangleleft}$ and $\star\,\mathbf{R}^{\blacktriangleright} = -i\mathbf{R}^{\blacktriangleright}$. In this case, the metrics are complex and solutions of the complexified Einstein equation.

[4] Recall that the Weyl tensor is $\mathbf{C}^{\rho\sigma}_{\mu\nu} = \mathbf{R}^{\rho\sigma}_{\mu\nu} - 2\mathbf{R}_{[\mu}{}^{[\rho}g_{\nu]}{}^{\sigma]} + \frac{1}{3}\mathbf{R}g_{[\mu}{}^{[\rho}g_{\nu]}{}^{\sigma]}$. The (anti-)self-duality is described in the Euclidean case:[5] $\star\mathbf{C}_{\mu\nu\rho\sigma} = (1/2)\varepsilon_{\mu\nu}{}^{\lambda\kappa}\mathbf{C}_{\lambda\kappa\rho\sigma} = \mathbf{C}_{\mu\nu\rho\sigma}$ and $\star\mathbf{C}_{\mu\nu\rho\sigma} = (1/2)\varepsilon_{\mu\nu}{}^{\lambda\kappa}\mathbf{C}_{\lambda\kappa\rho\sigma} = -\mathbf{C}_{\mu\nu\rho\sigma}$.

[6] The Einstein condition is equivalent to the metric $g_{\mu\nu}$ being solution of the Einstein vaccum solutions.

and $\mathcal{L}_{\text{Cosmological}}[e]$ which given by[7]:

$$
\begin{array}{rclrcl}
\mathcal{L}_{\text{EC}}[e,\omega] & \propto & \varepsilon_{IJKL} e^I \wedge e^J \wedge \mathbf{F}^{KL} & \mathcal{L}_{\text{Pontrjagin}}[e,\omega] & \propto & \mathbf{F}^{IJ} \wedge \mathbf{F}_{IJ} \\
\mathcal{L}_{\text{Holst}}[e,\omega] & \propto & e^I \wedge e^J \wedge \mathbf{F}_{IJ} & \mathcal{L}_{\text{Nieh-Yan}}[e,\omega] & \propto & \mathbf{T}^I \wedge \mathbf{T}_I - \mathbf{F}_{IJ} \wedge e^I \wedge e^J \\
\mathcal{L}_{\text{Euler}}[e,\omega] & \propto & \varepsilon_{IJKL} \mathbf{F}^{IJ} \wedge \mathbf{F}^{KL} & \mathcal{L}_{\text{Cosmological}}[e,\omega] & \propto & \varepsilon_{IJKL} e^I \wedge e^J \wedge e^K \wedge e^L
\end{array}
\tag{0.1}
$$

We will be interest in such a context with Schwarz type topological field theories: In 3D and 4D cases we have the Chern Simons Lagrangian, the Lagrangian for BF theory.[8]

Generalizations In the Einstein-Cartan action; we have $\mathcal{L}^{\text{3D}}_{\text{EH}} = \varepsilon_{IJK} e^I \wedge \mathbf{F}^{JK}$ and $\mathcal{L}^{\text{4D}}_{\text{EH}} = \varepsilon_{IJKL} e^I \wedge e^J \wedge \mathbf{F}^{KL}$ respectively for the 3D and 4D cases. What are the possible action in any dimension? The general principle of construction is built on the invariance under diffeomorphisms, and the local Lorentz invariance. The basic building block for first order gravity are the tetrad 1-form $e \in \Omega^1(\mathcal{X}, \mathbb{R}^{1,3})$ and the connection one form $\omega \in \Omega^1(\mathcal{X}, \mathfrak{g})$ - see Zanelli [54] [26]. More precisely, we consider the Lorentz invariant tensors 0-forms $\varepsilon_{I_1 \cdots I_n}$ and η_{IJ}, also e^I, ω^{IJ} which are a vector-valued 1-form $e \in \Omega^1(\mathcal{X}, \mathbb{R}^{1,3})$ and a connection 1-form, locally a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(\mathcal{X}, \mathfrak{g})$ and finally we consider also $\mathbf{T}^I, \mathbf{F}^{IJ}$ which are respectively the torsion 2-form $\mathbf{T} \in \Omega^2(\mathcal{X}, \mathbb{R}^{1,3})$ and the curvature 2-form, locally a $\mathfrak{g}$-valued 2-form $\mathbf{F} \in \Omega^2(\mathcal{X}, \mathfrak{g})$. Notice that the torsion is written:

$$
\begin{array}{rcl}
\mathbf{T}^I & = & \mathrm{d}e^I + \omega^I{}_J \wedge e^J \\
\mathbf{R}^{IJ} & = & \mathrm{d}\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ}
\end{array}
\tag{0.2}
$$

To summarize, we cite Zanelli [54]: *"We are interested in objects that transform in a controlled way under Lorentz rotations (vectors, tensors, spinors, etc.). The existence of Bianchi identities implies that differentiating these fields, the only tensors that can be produced are combinations of the same objects."* This movement lead us to consider the Lanczos-Lovelock series, with and without torsional implications. For example, the Lanczos-Lovelock (LL) Lagrangian[9] is described as $\mathcal{L}_{\text{LL}}[e,\omega] = \varpi_{LL} \int_{\boldsymbol{\mathcal{X}}} \sum_{k=0}^{n/2} \varpi_k \mathbf{L}_k^{(n)}$, see Zanelli [54] [26] for torsional series related to Lanczos-Lovelock (LL) Lagrangian and relation to topological field theory, topological invariant and characteristics classes.

1 Self-Duality and Gravity

1.1 Einstein-Cartan theory

We shall distinguish between three formulations. The first one is the Einstein-Hilbert formulation. In this approach the metric is the dynamical variable and satisfies Euler Lagrange equations. The fundamental objects - the Levi-Civita connection $\Gamma^\rho_{\mu\nu}$ and the curvature tensor $\mathbf{R}^\rho{}_{\mu\nu\sigma}$ - are expressed via the metric. In such a framework, we describe (GR) as a *metric theory.* The functional is

$$
\mathcal{L}_{\text{EH}}[g_{\mu\nu}] = \int_{\mathcal{X}} \boldsymbol{\mathcal{L}}[g_{\mu\nu}]\mathrm{d}\mathfrak{y} = \int_{\mathcal{X}} \mathbf{R}\sqrt{-g}\,\mathrm{d}\mathfrak{y} = \int_{\mathcal{X}} \sqrt{-g} g^{\mu\nu} \mathbf{R}_{\mu\nu}[g]\mathrm{d}\mathfrak{y}. \tag{1.1}
$$

[7] $\boldsymbol{\mathcal{L}}_{\text{Pontrjagin}}[\omega]$ is related to the second Chern class, or Pontryagin number. On the other side $\boldsymbol{\mathcal{L}}_{\text{Euler}}[\omega]$ is related to the Euler number.

[8] In 3D and 4D cases we have the Chern Simons Lagrangian, the Lagrangian for BF theory are respectively given by: $\mathcal{L}_{\text{CS}}(\mathbf{A}) = \frac{1}{2}\int_{\mathcal{X}} \langle \mathbf{A} \wedge \mathrm{d}\mathbf{A} + \frac{2}{3}\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} \rangle$ and $\mathcal{L}_{\text{BF}}[\mathbf{B},\mathbf{F}] = \frac{1}{2}\int_{\boldsymbol{\mathcal{X}}} \langle \mathbf{B} \wedge \mathbf{F} \rangle = \int_{\boldsymbol{\mathcal{X}}} \mathbf{B}^{IJ} \wedge \mathbf{F}_{IJ}$

[9] with $\mathbf{L}_k^{(n)} = \varepsilon_{I_1 \cdots I_n} \mathbf{F}^{I_1 I_2} \wedge \cdots \wedge \mathbf{F}^{I_{2k-1} I_k} \wedge e^{I_{2k+1}} \wedge \cdots \wedge e^{I_n}$

Then, the so-called the Einstein-Palatin setting or equivalently the Palatini first order form of the theory : we consider the connection and the metric as independent variables: $\mathcal{L}_{\mathsf{EH}}[g] \mapsto \mathcal{L}_{\mathsf{Palatini}}[g,\Gamma]$. Therefore, $\mathcal{L}_{\mathsf{Palatini}}[g,\Gamma]$ is written:

$$\mathcal{L}_{\mathsf{Palatini}}[g,\Gamma] = \int_{\mathcal{X}} \sqrt{-g} g^{\mu\nu} \mathbf{R}_{\mu\nu}[\Gamma] \mathrm{d}\mathfrak{y}. \tag{1.2}$$

We perform respectively the variations $\delta\Gamma$ and δg. The variations with respect to the former leads to set the connection Γ to be the Levi-Civita affine connection, while variations with respect to the latter give the Einstein vacuum equations. Finally, the last one is the Einstein-Palatini action $\mathcal{L}_{\mathsf{EH}}[e,\omega] = \int \varepsilon_{IJKL} e^I \wedge e^J \wedge \mathbf{F}^{KL}$ (or Einstein-Cartan action) involved the following canonical variables: the spin connection ω and the vielbein, or tetrad field e. The Einstein equations writes:

$$\left| \begin{array}{rcl} \mathrm{d}_\omega e^I = \mathrm{d}e^I + \omega^I{}_J \wedge e^J & = & 0 \\ \varepsilon_{IJKL} e^J \wedge \mathbf{F}^{KL} & = & 0 \end{array} \right. \tag{1.3}$$

1.2 Hodge duality on a vector space

1.2.1 $\mathcal{V}$-valued p-forms $\Omega^p(\mathcal{X},\mathcal{V})$

We consider $\mathcal{V}$-valued p-forms $\Omega^p(\mathcal{X},\mathcal{V})$, those n-forms with values in a vector space $\mathcal{V}$. Then we write $\varphi \in \Omega^p(\mathcal{X},\mathcal{V})$. In the following sections, we will be interested in forms of different types, depending on the nature of the vector bundle. It may be the associated bundle $\mathcal{P} \times_\rho \mathcal{V}$ (see below), a Lie algebra $\mathfrak{g}$ and finally we can consider the vector space where the forms take values in $\mathcal{V} \otimes \mathcal{V}^\star \cong \mathrm{End}\mathcal{V}$.[10] First, we consider the product of differential forms $\boldsymbol{\varphi} \in \Omega^p(\mathcal{X},\mathfrak{g})$ and $\boldsymbol{\psi} \in \Omega^q(\mathcal{X},\mathcal{V})$. This one is given[11] by $(\rho_\mathfrak{g})_\wedge = \rho_\mathfrak{g}(\boldsymbol{\varphi}) \wedge \boldsymbol{\psi} \in \Omega^{p+q}(\mathcal{X},\mathcal{V})$ along with the following rule. [1] Let $\zeta_1, ..., \zeta_{p+q}$ vectors fields on $\mathcal{X}$,

$$\rho_\mathfrak{g}(\boldsymbol{\varphi}) \wedge \boldsymbol{\psi}(\zeta_1 ... \zeta_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathcal{S}_{p+q}} (-1)^{\boldsymbol{\sigma}} \rho_\mathfrak{g}(\boldsymbol{\varphi}(\zeta_{\sigma(1)} ... \zeta_{\sigma(p)}))(\boldsymbol{\psi}(\zeta_{\sigma(p+1)} ... \zeta_{\sigma(p+q)})) \tag{1.5}$$

The product introduce in the following (1.8) allows us to picture $\Omega^\star(\mathcal{X},\mathfrak{g})$ as a graded Lie algebra structure with the bracket $[\![,]\!]$. Then, for any $\boldsymbol{\varphi} \in \Omega^p(\mathcal{X},\mathfrak{g})$ and $\boldsymbol{\psi} \in \Omega^q(\mathcal{X},\mathfrak{g})$, we define

$$[\![\boldsymbol{\varphi},\boldsymbol{\psi}]\!](\zeta_1 ... \zeta_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathcal{S}_{p+q}} (-1)^{\boldsymbol{\sigma}} [(\boldsymbol{\varphi}(\zeta_{\sigma(1)} ... \zeta_{\sigma(p)})), (\boldsymbol{\psi}(\zeta_{\sigma(p+1)} ... \zeta_{\sigma(p+q)}))]_\mathfrak{g} \tag{1.6}$$

Finally, the last case of interest is when $\boldsymbol{\varphi} \in \Omega^n(\mathcal{X}, \mathrm{End}(\mathcal{V})) = \Omega^n(\mathcal{X}, \mathcal{V} \otimes \mathcal{V}^\star)$. For that purpose, considering the tensor algebra generated by $\mathcal{V}$, such forms falls in the category of form in $\Omega^\star(\mathcal{X}, \bigotimes \mathcal{V})$ for $\boldsymbol{\varphi} \in \Omega^\star(\mathcal{X}, \bigotimes \mathcal{V})$ and $\boldsymbol{\psi} \in \Omega^\star(\mathcal{X}, \bigotimes \mathcal{V})$. We define the associative bigraded product :

$$\begin{aligned} \boldsymbol{\varphi} \wedge \boldsymbol{\psi}(\zeta_1 ... \zeta_{p+q}) &= (\boldsymbol{\varphi} \otimes_\wedge \boldsymbol{\psi})(\zeta_1 ... \zeta_{p+q}) \\ &= \frac{1}{p!q!} \sum_{\sigma \in \mathcal{S}_{p+q}} (-1)^{\boldsymbol{\sigma}} (\boldsymbol{\varphi}(\zeta_{\sigma(1)} ... \zeta_{\sigma(p)})) \otimes (\boldsymbol{\psi}(\zeta_{\sigma(p+1)} ... \zeta_{\sigma(p+q)})) \end{aligned} \tag{1.7}$$

[10] We denote $\mathrm{End}(\mathcal{V})$ the vector space of endomorphism of $\mathcal{V}$. Canonically we identify $\mathcal{V} \otimes \mathcal{V}^\star \cong \mathrm{End}(\mathcal{V})$. Any element $\Xi = \Xi^I_J \mathbf{v}_I \otimes \mathbf{v}^J \in \mathcal{V} \otimes \mathcal{V}^\star$ is identified with the endomorphism $\Xi_\mathcal{V} \in \mathrm{End}\mathcal{V}$ described as follow :

$$\Xi_\mathcal{V} : v = v^I \mathbf{v}_I \mapsto \Xi_\mathcal{V}(v) = \Xi v = \Xi^I_J \mathbf{v}_I \mathbf{v}^J(v) = (\Xi^I_J v^J) \mathbf{v}_I \tag{1.4}$$

Hence, any endomorphism $\Xi_\mathcal{V} \in \mathrm{End}\mathcal{V}$ is given by the matrix Ξ^I_J - in the basis $\mathbf{v}_I$ of the vector space $\mathcal{V}$.

[11] Following [1] we notice that $(\rho_\mathfrak{g})_\wedge : \Omega^\star(\mathcal{X},\mathcal{V}) \longrightarrow \Omega^{\star+p}(\mathcal{X},\mathcal{V})$ is a graded $\Omega(\mathcal{X})$-module homomorphism of degree p.

1.2.2 $\mathfrak{g}$-valued n-forms $\Omega^p(\mathcal{X},\mathfrak{g})$

Let λ be a $\mathfrak{g}$-valued p-form on $\mathcal{X}$ (*i.e* $\lambda \in \Omega^p(\mathcal{X},\mathfrak{g}) = \Omega^p(\mathcal{X}) \otimes \mathfrak{g}$). Let $\mathfrak{b}_{\mathcal{I}}$ be a basis on $\mathfrak{g}$. Now, $\forall \lambda \in \Omega^p(\mathcal{X},\mathfrak{g})$ writes as[12] : $\lambda = \lambda^{\mathcal{I}} \otimes \mathfrak{b}_{\mathcal{I}}$. If we denote another $\mathfrak{g}$-valued p-form on $\mathcal{X}$ *i.e* $\sigma = \sigma^{\mathcal{J}} \otimes \mathfrak{b}_{\mathcal{J}} \in \Omega^q(\mathcal{X},\mathfrak{g}) = \Omega^q(\mathcal{X}) \otimes \mathfrak{g}$. The bracket of λ, σ denoted as $[\lambda,\sigma]$

$$[\lambda,\sigma] = (\lambda^{\mathcal{I}} \wedge \sigma^{\mathcal{J}}) \otimes [\mathfrak{b}_{\mathcal{I}}, \mathfrak{b}_J] = \mathfrak{c}^{\mathcal{K}}_{\mathcal{I}\mathcal{J}}(\lambda^{\mathcal{I}} \wedge \sigma^{\mathcal{J}}) \otimes \mathfrak{b}_{\mathcal{K}} \tag{1.8}$$

where $\mathfrak{c}^{\mathcal{K}}_{\mathcal{I}\mathcal{J}}$ are the structures constants of the Lie algebra. We use the wedge product on the form part and the classical Lie bracket on the Lie algebra part. We observe the following graded property (1.9)(i) and the graded Jacobi identity (1.9)(ii)

$$\text{(i)} \quad [\lambda,\sigma] = (-1)^{pq+1}[\sigma,\lambda] \qquad \text{(ii)} \quad [\lambda,[\sigma,\eta]] = [[\lambda,\sigma]\eta] + (-1)^{pq}[\sigma,[\lambda,\eta]] \tag{1.9}$$

Let $\omega \in \Omega^1(\mathcal{X},\mathfrak{g})$ be a $\mathfrak{g}$-valued 1-form and $\zeta_1,\zeta_2 \in \mathfrak{X}(\mathcal{X})$ - Here, $\mathfrak{X}(\mathcal{X})$ denote the set of vector fields on $\mathcal{X}$ we have[13]

$$[\omega,\omega](\zeta_1,\zeta_2) = 2[\omega(\zeta_1),\omega(\zeta_2)] \tag{1.12}$$

The previous lines are actually similar to a more general n-form with value in the vector space $\mathcal{V}$, $\lambda \in \Omega^n(\mathcal{X},\mathcal{V})$. Here, we prefer to directly consider the specific case of $\mathcal{V}$ being a Lie-algebra, allowing us to write the expression of the bracket with (1.13). Let $\lambda \in \Omega^p(\mathcal{X},\mathfrak{g})$ and $\sigma \in \Omega^q(\mathcal{X},\mathfrak{g})$, we obtain:

$$[\lambda,\sigma] = \lambda^{\mathcal{I}} \wedge \sigma^{\mathcal{J}} \otimes [\mathfrak{b}_{\mathcal{I}}, \mathfrak{b}_{\mathcal{J}}] = \lambda \wedge \sigma - (-1)^{pq}\sigma \wedge \lambda \tag{1.13}$$

When $\mathfrak{g}$ is a matrix Lie algebra[14] we use the notation $\lambda \wedge \sigma = \mathfrak{b}_{\mathcal{I}}\mathfrak{b}_{\mathcal{J}} \otimes \lambda^{\mathcal{I}} \wedge \lambda^{\mathcal{J}}$. Therefore, for any odd degree n, let $\lambda \in \Omega^n(\mathcal{X},\mathfrak{g})$, we find $[\lambda,\lambda] = 2\lambda \wedge \lambda$. In the case of matrix Lie algebra we often find this notation.

1.2.3 Hodge star operator: Basis-independent formulas

We consider an oriented Riemannian 4D-manifold $(\mathcal{X},g)$. Recall that the Hodge star operator is described as:

$$\star : \begin{cases} \Omega^p(\mathcal{X}) & \to & \Omega^{n-p}(\mathcal{X}) \\ \varphi & \mapsto & \star\varphi \end{cases}$$

We define for each $m \in \mathcal{X}$, with $\eta = \eta_\mu \mathrm{d}x^\mu$ and $\rho = \rho_\nu \mathrm{d}x^\nu$, the inner product on $T^\star_x\mathcal{X}$

$$\langle \rho,\eta \rangle_{\Omega^1(\mathcal{X})} = \sum_{\mu,\nu} g^{\mu\nu}\eta_\mu\rho_\nu \qquad \Longleftrightarrow \qquad \langle \rho,\eta \rangle_{\Omega^1(\mathcal{X})}(x) = \sum_{\mu,\nu} g^{\mu\nu}(x)\eta_\mu(x)\rho_\nu(x)$$

[12]we use the curved capital letters to underlined the fact that we work with a basis of generator of the Lie algebra.

[13]⌈ Following [32] we decompose $\omega = \omega^I \otimes \mathfrak{b}_{\mathcal{I}}$

$$\begin{aligned} [\omega,\omega](\zeta_1,\zeta_2) &= [\omega^I \otimes \mathfrak{b}_{\mathcal{I}}, \omega^J \otimes \mathfrak{b}_{\mathcal{J}}](\zeta_1,\zeta_2) = \omega^I \wedge \omega^{\mathcal{J}} \otimes [\mathfrak{b}_{\mathcal{I}}, \mathfrak{b}_{\mathcal{J}}](\zeta_1,\zeta_2) = \mathfrak{c}^{\mathcal{K}}_{\mathcal{I}\mathcal{J}}(\omega^{\mathcal{I}} \wedge \omega^{\mathcal{J}}) \otimes \mathfrak{b}_{\mathcal{K}}(\zeta_1,\zeta_2) \\ &= \left(\mathfrak{c}^{\mathcal{K}}_{\mathcal{I}\mathcal{J}}(\omega^{\mathcal{I}} \otimes \omega^{\mathcal{J}} - \omega^{\mathcal{J}} \otimes \omega^{\mathcal{I}}) \otimes \mathfrak{b}_{\mathcal{K}}\right)(\zeta_1,\zeta_2) = 2\left(\mathfrak{c}^{\mathcal{K}}_{\mathcal{I}\mathcal{J}}\omega^{\mathcal{I}}(\zeta_1)\omega^{\mathcal{J}}(\zeta_2)\right) \otimes \mathfrak{b}_{\mathcal{K}} \\ &= 2\mathfrak{c}^{\mathcal{K}}_{\mathcal{I}\mathcal{J}}\omega^{\mathcal{I}}(\zeta_1)\omega^{\mathcal{J}}(\zeta_2)\mathfrak{b}_{\mathcal{K}} \end{aligned} \tag{1.10}$$

On the other hand, let $\lambda,\sigma \in \Omega^1(\mathcal{X},\mathfrak{g})$ and $\zeta_1,\zeta_2 \in \mathfrak{X}(\mathcal{X})$. We define $[\lambda,\sigma](\zeta_1,\zeta_2) = [\lambda(\zeta_1),\sigma(\zeta_2)]$ then :

$$[\omega,\omega](\zeta_1,\zeta_2) = [\omega(\zeta_1),\omega(\zeta_2)] = [\omega^{\mathcal{I}}(\zeta_1) \otimes \mathfrak{b}_{\mathcal{I}}, \omega^{\mathcal{J}}(\zeta_2) \otimes \mathfrak{b}_{\mathcal{J}}] = \omega^{\mathcal{I}}(\zeta_1)\omega^{\mathcal{J}}(\zeta_2)[\mathfrak{b}_{\mathcal{I}}, \mathfrak{b}_{\mathcal{J}}] \tag{1.11}$$

Hence, comparing (**??**) and (1.11) we obtain (1.12).

[14]This relation, is generally given for $\mathcal{V}$ being an associative algebra.

and the inner product on $\Lambda^k T_x^\star \mathcal{X}$. Let consider $\boldsymbol{\eta} = (1/k!)\boldsymbol{\eta}_{\mu_1\cdots\mu_k} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k} \in \Omega^k(\mathcal{X})$ and $\boldsymbol{\rho} = (1/k!)\boldsymbol{\rho}_{\mu_1\cdots\mu_k} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k} \in \Omega^k(\mathcal{X})$ is given by

$$\langle \boldsymbol{\rho}, \boldsymbol{\eta} \rangle_{\Omega^k(\mathcal{X})} = \langle \frac{1}{k!}\boldsymbol{\eta}_{\mu_1\cdots\mu_k} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k}, \frac{1}{k!}\boldsymbol{\rho}_{\nu_1\cdots\nu_k} \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_k} \rangle_{\Omega^k(\mathcal{X})}$$

which in turns is written:

$$\langle \boldsymbol{\rho}, \boldsymbol{\eta} \rangle_{\Omega^k(\mathcal{X})} = \frac{1}{k!} \sum_{\mu_1,\cdots,\mu_k} \sum_{\nu_1,\cdots,\nu_k} g^{\mu_1\nu_1} \cdots g^{\mu_k\nu_k} \boldsymbol{\eta}_{\mu_1\cdots\mu_k} \boldsymbol{\rho}_{\nu_1\cdots\nu_k} \tag{1.14}$$

We denote by $\mathrm{d}\mathfrak{y} = \mathrm{d}\mathfrak{y}_g$ the associated Riemanian volume form. The alternative definition of the Hodge star operator $\star : \Omega^k(\mathcal{X}) \to \Omega^{n-k}(\mathcal{X})$ maps any k-form $\boldsymbol{\eta} \in \Omega^k(\mathcal{X})$ to $\star\boldsymbol{\eta}$, where $\star\boldsymbol{\eta} \in \Omega^{n-k}(\mathcal{X})$ such that, $\forall \boldsymbol{\rho} \in \Omega^k(\mathcal{X})$:

$$\boldsymbol{\rho} \wedge \star\boldsymbol{\eta} = \langle \boldsymbol{\rho}, \boldsymbol{\eta} \rangle \mathrm{d}\mathfrak{y}_g$$

We have the following properties:

$$\left| \begin{array}{lcl} \boldsymbol{\eta} \wedge \star\boldsymbol{\rho} & = & \boldsymbol{\rho} \wedge \star\boldsymbol{\eta} \\ \boldsymbol{\rho} \wedge \star\boldsymbol{\eta} & = & \langle \boldsymbol{\rho}, \boldsymbol{\eta} \rangle \mathrm{d}\mathfrak{y}_g \\ \langle \boldsymbol{\rho}, \boldsymbol{\eta} \rangle & = & \langle \star\boldsymbol{\rho}, \star\boldsymbol{\eta} \rangle \end{array} \right. \tag{1.15}$$

and in the Riemaniann case: $\left| \begin{array}{lcl} \star 1 & = & \mathrm{d}\mathfrak{y}_g \\ \star \mathrm{d}\mathfrak{y}_g & = & 1 \end{array} \right.$. We have the important following fact, for any $\boldsymbol{\eta} \in \Omega^k(\mathcal{X})$

$$\star\star\boldsymbol{\eta} = (-1)^{\boldsymbol{\sigma}+k(n-k)}\boldsymbol{\eta} \tag{1.16}$$

where $\boldsymbol{\sigma}$ is the number of negative values, or timelike directions.

$$\left| \begin{array}{ll} \text{Riemannian } (\boldsymbol{\sigma} = 0) & \star\star\boldsymbol{\eta} = (-1)^{k(n-k)}\boldsymbol{\eta} = (-1)^{k(n+1)}\boldsymbol{\eta} \\ \text{Lorentzian } (\boldsymbol{\sigma} = 1) & \star\star\boldsymbol{\eta} = (-1)^{1+k(n-k)}\boldsymbol{\eta} = (-1)^{1+k(n+1)}\boldsymbol{\eta} \end{array} \right.$$

Then we have an isomorphism $\Omega^k(\mathcal{X}) \cong \Omega^{n-k}(\mathcal{X})$. We define the inverse of $\star$, denoted $\star^{-1}$ which in turn is written

$$\left| \begin{array}{ll} \text{Riemannian } (\boldsymbol{\sigma} = 0) & \star^{-1} = (-1)^{k(n-k)}\star \\ \text{Lorentzian } (\boldsymbol{\sigma} = 1) & \star^{-1} = (-1)^{1+k(n-k)}\star \end{array} \right.$$

which is generally given by: $\star^{-1} = (-1)^{\boldsymbol{\sigma}}(-1)^{k(n-k)}\star$. We define the global inner product (bilinear Hodge L^2 inner product)

$$\langle \boldsymbol{\eta}, \boldsymbol{\rho} \rangle = \int_{\mathcal{X}} \langle \boldsymbol{\eta}, \boldsymbol{\rho} \rangle_{\Omega^k(\mathcal{X})} \mathrm{d}\mathfrak{y} = \int_{\mathcal{X}} \boldsymbol{\eta} \wedge \star\boldsymbol{\rho} \tag{1.17}$$

Since, $\boldsymbol{\eta} \wedge \star\boldsymbol{\rho} = \boldsymbol{\rho} \wedge \star\boldsymbol{\eta}$ the product $\langle , \rangle$ is symmetric $\langle \boldsymbol{\eta}, \boldsymbol{\rho} \rangle = \int_{\mathcal{X}} \boldsymbol{\eta} \wedge \star\boldsymbol{\rho} = \int_{\mathcal{X}} \boldsymbol{\rho} \wedge \star\boldsymbol{\eta} = \langle \boldsymbol{\rho}, \boldsymbol{\eta} \rangle$. We define the *co-differential* of $\boldsymbol{\eta} \in \Omega^k(\mathcal{X})$, denoted $\delta\boldsymbol{\eta} \in \Omega^{k-1}(\mathcal{X})$ such that for any $\boldsymbol{\rho} \in \Omega^{k+1}(\mathcal{X})$ $\langle \boldsymbol{\eta}, \mathrm{d}\boldsymbol{\rho} \rangle = \langle \delta\boldsymbol{\eta}, \boldsymbol{\rho} \rangle$ which is equivalently written on any $\boldsymbol{\eta} \in \Omega^k(\mathcal{X})$, gives $\delta\boldsymbol{\eta} \in \Omega^{k-1}(\mathcal{X})$

$$\left| \begin{array}{ll} (\mathcal{X}, g) \text{ Riemannian Case} & \delta\boldsymbol{\eta} = (-1)^{n(k+1)+1} \star \mathrm{d} \star \boldsymbol{\eta} \\ (\mathcal{X}, g) \text{ Lorentzian Case} & \delta\boldsymbol{\eta} = (-1)^{n(k+1)} \star \mathrm{d} \star \boldsymbol{\eta} \end{array} \right.$$

If we consider a general pseudo-Riemanian metric of type $\boldsymbol{\sigma}$, we obtain:

$$\delta\boldsymbol{\eta} = (-1)^{\boldsymbol{\sigma}}(-1)^{n(k+1)+1} \star \mathrm{d} \star \boldsymbol{\eta} \tag{1.18}$$

so that

$$\begin{aligned} \langle \boldsymbol{\eta}, \mathrm{d}\boldsymbol{\rho} \rangle &= \langle \mathrm{d}\boldsymbol{\rho}, \boldsymbol{\eta} \rangle = \int_{\mathcal{X}} \mathrm{d}\boldsymbol{\rho} \wedge \star\boldsymbol{\eta} = \int_{\mathcal{X}} \left[\mathrm{d}(\boldsymbol{\rho} \wedge \star\boldsymbol{\eta}) - (-1)^{k-1} \boldsymbol{\rho} \wedge \mathrm{d}(\star\boldsymbol{\eta}) \right] \\ &= (-1)^k \int_{\mathcal{X}} \boldsymbol{\rho} \wedge \mathrm{d}(\star\boldsymbol{\eta}) \end{aligned} \tag{1.19}$$

on the other side:

$$\begin{aligned} \langle \delta\boldsymbol{\eta}, \boldsymbol{\rho} \rangle &= \langle (-1)^{n(k+1)+1+\boldsymbol{\sigma}} \star \mathrm{d} \star \boldsymbol{\eta}, \boldsymbol{\rho} \rangle = \langle \boldsymbol{\rho}, (-1)^{n(k+1)+1+\boldsymbol{\sigma}} \star \mathrm{d} \star \boldsymbol{\eta} \rangle \\ &= \int_{\mathcal{X}} \boldsymbol{\rho} \wedge \star \big[(-1)^{n(k+1)+1+\boldsymbol{\sigma}} \star \mathrm{d} \star \boldsymbol{\eta} \big] = \int_{\mathcal{X}} (-1)^{n(k+1)+1+\boldsymbol{\sigma}} \int_{\mathcal{X}} \boldsymbol{\rho} \wedge \star\star \left[\mathrm{d} \star \boldsymbol{\eta} \right] \end{aligned}$$

But since: $\mathrm{d} \star \boldsymbol{\eta}$ is a $\big[(n-k)+1\big]$-form

$$\star\star \left[\mathrm{d} \star \boldsymbol{\eta} \right] = (-1)^{\left[(n-k)+1\right]\left[n - \left[(n-k)+1\right]\right]} \left[\mathrm{d} \star \boldsymbol{\eta} \right] = (-1)^{\left[(n-k)+1\right]\left[k-1\right]} \left[\mathrm{d} \star \boldsymbol{\eta} \right]$$

so that:

$$\langle \delta\boldsymbol{\eta}, \boldsymbol{\rho} \rangle = \underbrace{(-1)^{n(k+1)+1+\boldsymbol{\sigma}} (-1)^{\left[(n-k)+1\right]\left[k-1\right]}}_{(-1)^k} \int_{\mathcal{X}} \boldsymbol{\rho} \wedge \left[\mathrm{d} \star \boldsymbol{\eta} \right] \tag{1.20}$$

We conclude from (1.19) and (1.20) that $\langle \boldsymbol{\eta}, \mathrm{d}\boldsymbol{\rho} \rangle = \langle \delta\boldsymbol{\eta}, \boldsymbol{\rho} \rangle$.

1.2.4 Hodge Laplacian, Poincaré duality, Hodge decomposition

The Hodge Laplacian $\boldsymbol{\Delta}$ is the operator defined by:[15]

$$\boldsymbol{\Delta} : \begin{cases} \Omega^p(\mathcal{X}) & \to \quad \Omega^p(\mathcal{X}) \\ \boldsymbol{\eta} & \mapsto \quad \boldsymbol{\Delta}\boldsymbol{\eta} = \mathrm{d}\delta\boldsymbol{\eta} + \delta\mathrm{d}\boldsymbol{\eta} \end{cases}$$

Note that: $\langle \boldsymbol{\eta}, \boldsymbol{\Delta}\boldsymbol{\rho} \rangle = \langle \boldsymbol{\eta}, \mathrm{d}\delta\boldsymbol{\rho} \rangle + \langle \boldsymbol{\eta}, \delta\mathrm{d}\boldsymbol{\rho} \rangle = \langle \delta\boldsymbol{\eta}, \delta\boldsymbol{\rho} \rangle + \langle \mathrm{d}\boldsymbol{\eta}, \mathrm{d}\boldsymbol{\rho} \rangle$. So that $\boldsymbol{\Delta}$ is self-adjoint $\langle \boldsymbol{\eta}, \boldsymbol{\Delta}\boldsymbol{\rho} \rangle = \langle \boldsymbol{\Delta}\boldsymbol{\eta}, \boldsymbol{\rho} \rangle$ and we observe $\langle \boldsymbol{\Delta}\boldsymbol{\eta}, \boldsymbol{\eta} \rangle = \mid \delta\boldsymbol{\eta} \mid^2 + \mid \mathrm{d}\boldsymbol{\eta} \mid^2 \neq 0$. These considerations leads to the notion of a harmonic k-form is such that $\boldsymbol{\Delta}\boldsymbol{\omega} = 0$ which is equivalent to $\boldsymbol{\Delta}\boldsymbol{\omega} = 0 \iff \mathrm{d}\boldsymbol{\omega} = \mathrm{d} \star \boldsymbol{\omega} = 0$. Indeed[16], if $\mathrm{d}\boldsymbol{\omega} = \mathrm{d} \star \boldsymbol{\omega} = 0$, then,

$$\boldsymbol{\Delta}\boldsymbol{\omega} = \mathrm{d}\delta\boldsymbol{\omega} + \delta\mathrm{d}\boldsymbol{\omega} = \mathrm{d}\delta\boldsymbol{\omega} = \mathrm{d}\big[\boldsymbol{\sigma}(-1)^{n(k+1)+1} \star \mathrm{d} \star \boldsymbol{\omega}\big] = 0$$

We consider the three subspaces $\mathrm{d}\Omega^{k-1}(\mathcal{X})$, $\delta\Omega^{k+1}(\mathcal{X})$ and $\Omega^k_{\mathsf{Harm.}}(\mathcal{X})$ in $\Omega^k\mathcal{X}$, if $\boldsymbol{\eta} \in \mathrm{d}\Omega^{k-1}(\mathcal{X})$, $\boldsymbol{\rho} \in \delta\Omega^{k+1}(\mathcal{X})$, $\boldsymbol{\omega} \in \Omega^k_{\mathsf{Harm.}}(\mathcal{X})$

$$\begin{aligned} \langle \mathrm{d}\boldsymbol{\eta}, \delta\boldsymbol{\rho} \rangle &= \langle \mathrm{d}\mathrm{d}\boldsymbol{\eta}, \boldsymbol{\rho} \rangle = 0 \\ \langle \mathrm{d}\boldsymbol{\eta}, \boldsymbol{\omega} \rangle &= \langle \boldsymbol{\eta}, \delta\boldsymbol{\omega} \rangle = 0 \\ \langle \delta\boldsymbol{\rho}, \boldsymbol{\omega} \rangle &= \langle \boldsymbol{\rho}, \mathrm{d}\boldsymbol{\omega} \rangle = 0 \end{aligned}$$

[15] Equivalently we write: $\boldsymbol{\Delta} = \left[\mathrm{d} + \delta\right]^2 = \mathrm{d} \circ \left[\boldsymbol{\sigma}(-1)^{n(k+1)+1} \star \mathrm{d} \star\right] + \left[\boldsymbol{\sigma}(-1)^{n(k+1)+1} \star \mathrm{d} \star\right] \circ \mathrm{d}$

[16] It remain to prove the equivalence in the other direction

1.2.5 Hodge star operator: component formula

We now explore the relation between Hodge star operator and the Einstein-Cartan equations. First we recall some basic properties about Hodge duality.

Definition 1.1. *Let $\mathcal{V}$ be a n-dimensional vector space with inner product $\langle,\rangle_{\mathcal{V}}$ of signature $(n-q,q)$. We now consider the Hodge dual in $\Lambda\mathcal{V}$ denoted as $\star_{\mathcal{V}}=\star$. Now we consider $\{\theta_i\}$ an ordered basis for the vector space $\mathcal{V}$ then a p-form $\boldsymbol{\varphi}\in\Lambda^p\mathcal{V}$ is written as*

$$\boldsymbol{\varphi}=\frac{1}{p!}\boldsymbol{\varphi}_{i_1\dots i_p}\theta^{i_1}\wedge\dots\wedge\theta^{i_p}$$

then the Hodge dual $\star\boldsymbol{\varphi}\in\Lambda^{n-p}\mathcal{V}$ is given by (1.21) :

$$\star\boldsymbol{\varphi}=\frac{1}{(n-p)!}\big(\star\boldsymbol{\varphi}_{j_1\dots j_{n-p}}\big)\theta^{j_1}\wedge\dots\wedge\theta^{j_{(n-p)}} \tag{1.21}$$

We denote the components of $\star\boldsymbol{\varphi}$ in the basis $\theta^{j_1}\wedge\dots\wedge\theta^{j_{(n-p)}}$ by $\big(\star\boldsymbol{\varphi}_{j_1\dots j_{n-p}}\big)=\big(\star\boldsymbol{\varphi}\big)_{j_1\dots j_{n-p}}$. Then, we obtain (1.22) the expression :

$$\star\boldsymbol{\varphi}_{j_1\dots j_{n-p}}=\frac{1}{p!}\varepsilon^{i_1\dots i_p}{}_{j_1\dots j_{n-p}}\boldsymbol{\varphi}_{i_1\dots i_p} \tag{1.22}$$

From definition (1.1), we consider the case where the considered vector space is the Riemanian space-time : $\mathcal{V}=\mathcal{X}$. Let $\{\theta^\mu\}=\{\mathrm{d}x^\mu\}$ be a holonomic basis, and let $\forall\boldsymbol{\varphi}\in\Lambda^p\mathcal{X}$ be a p-form on $\mathcal{X}$ we write it as : $\boldsymbol{\varphi}=(1/p!)\boldsymbol{\varphi}_{i_1\dots i_p}dx^{i_1}\wedge\dots\wedge dx^{i_p}$. We consider the Hodge dual in $\Lambda\mathcal{X}$, written $\star\boldsymbol{\varphi}\in\Lambda^{n-p}\mathcal{X}$ and given by :

$$\star\boldsymbol{\varphi}=\frac{1}{(n-p)!}\big(\star\boldsymbol{\varphi}_{\mu_1\dots\mu_{n-p}}\big)\mathrm{d}x^{\mu_1}\wedge\dots\wedge\mathrm{d}x^{\mu_{n-p}} \tag{1.23}$$

with $\big(\star\boldsymbol{\varphi}_{\mu_1\dots\mu_{n-p}}\big)=\dfrac{1}{p!}\boldsymbol{\varepsilon}_{\mu_1\dots\mu_n}\boldsymbol{\varphi}^{\mu_1\cdot\cdot\mu_p}$. We notice that $\boldsymbol{\varphi}^{\mu_1\cdots\mu_p}=g^{\mu_1\nu_1}\dots g^{\mu_p\nu_p}\boldsymbol{\varphi}_{\nu_1\dots\nu_p}$ is the contravariant object and we raise and down indices when one contracts with the *metric tensor*.

⌈ We write $\star\mathbf{R}=\star(\theta^\mu\wedge\theta^\nu)\wedge\mathbf{R}_{\mu\nu}$. The purpose here is to derive Einstein equation thanks to the use of the external Hodge operator. Let us denote $\mathfrak{U}^{\mu\nu}=(\theta^\mu\wedge\theta^\nu)$. In fact, more generally for any p-form we denote: $\mathfrak{U}^{\mu_1\cdots\mu_p}=\bigwedge\limits_{1\le i\le p}\theta^{\mu_i}$. In the case of 4D Riemannian manifold,

$$\left|\begin{array}{lcl}\mathfrak{U} & = & 1\\ \mathfrak{U}^{\mu} & = & \theta^\mu\\ \mathfrak{U}^{\mu\nu} & = & \theta^\mu\wedge\theta^\nu\\ \mathfrak{U}^{\mu\nu\rho} & = & \theta^\mu\wedge\theta^\nu\wedge\theta^\rho\\ \mathfrak{U}^{\mu\nu\rho\sigma} & = & \theta^\mu\wedge\theta^\nu\wedge\theta^\rho\wedge\theta^\sigma\end{array}\right. \tag{1.24}$$

First, we demonstrate that $\star\mathbf{R}=\mathbf{R}\mathrm{d}\mathfrak{y}=\star\mathfrak{U}_{\mu\nu}\wedge\mathbf{R}^{\mu\nu}$, since:

$$\star\mathfrak{U}^{\mu\nu}=\star(\theta^\mu\wedge\theta^\nu)=\frac{\sqrt{-g}}{2!}\varepsilon_{\alpha\beta\rho\sigma}g^{\alpha\mu}g^{\beta\nu}\theta^\rho\wedge\theta^\sigma=\frac{1}{2}\boldsymbol{\varepsilon}_{\alpha\beta\rho\sigma}g^{\alpha\mu}g^{\beta\nu}\theta^\rho\wedge\theta^\sigma$$

Therefore we express the covariant quantity $\star\mathfrak{U}_{\alpha\beta}$, contracting two time with the metric tensor:

$$\star\mathfrak{U}_{\alpha\beta}=g_{\alpha\mu}g_{\beta\nu}\star\mathfrak{U}^{\mu\nu}=\frac{1}{2}\boldsymbol{\varepsilon}_{\alpha\beta\rho\sigma}\theta^\rho\wedge\theta^\sigma \tag{1.25}$$

Before going one step further, let focus on calculations beyond $vol(g)\mathbf{R} = \mathrm{d}\mathfrak{y}\sqrt{-g}\mathbf{R}$. We hove :

⌈ Let us evaluate $vol(g)\mathbf{R} = \mathrm{d}\mathfrak{y}\sqrt{-g}\mathbf{R}$, the integrand of the Einstein Hilbert action. We have, contracting the Riemannn curvature tensor the following equality $\mathbf{R} = \mathbf{R}^{\alpha\beta}{}_{\rho\sigma}\delta^{\rho}_{[\alpha}\delta^{\sigma}_{\beta]}$. Therefore,

$$vol(g)\mathbf{R} = vol(g)\delta^{\rho}_{[\alpha}\delta^{\sigma}_{\beta]}\mathbf{R}^{\alpha\beta}{}_{\rho\sigma} = \frac{1}{4}vol(g)(-1)^s\boldsymbol{\varepsilon}_{\mu\nu\alpha\beta}\boldsymbol{\varepsilon}^{\mu\nu\rho\sigma}\mathbf{R}^{\alpha\beta}{}_{\rho\sigma}$$

We have used (with $n = 4$ and p = 2) the following relation $\delta^{\rho}_{[\alpha}\delta^{\sigma}_{\beta]}p!(n-p)!(-1)^{\boldsymbol{\sigma}} = \boldsymbol{\varepsilon}_{\mu\nu\alpha\beta}\boldsymbol{\varepsilon}^{\mu\nu\rho\sigma}$. Then, expanding the volume form written in a holonomic coframe $\{\theta^\mu\}$, we get :

$$vol(g) = \sqrt{-g}\theta^0\wedge\theta^1\wedge\theta^2\wedge\theta^3 = \frac{\sqrt{-g}}{4!}\varepsilon_{\lambda\kappa\tau\gamma}\theta^\lambda\wedge\theta^\kappa\wedge\theta^\tau\wedge\theta^\gamma = \frac{1}{4!}\boldsymbol{\varepsilon}_{\lambda\kappa\tau\gamma}\theta^\lambda\wedge\theta^\kappa\wedge\theta^\tau\wedge\theta^\gamma$$

Since $\boldsymbol{\varepsilon}^{\mu\nu\rho\sigma}\boldsymbol{\varepsilon}_{\lambda\kappa\tau\gamma} = (-1)^s 4!\delta^{[\mu}_{\lambda}\delta^{\nu}_{\kappa}\delta^{\rho}_{\tau}\delta^{\sigma]}_{\gamma}$:

$$\begin{aligned} vol(g)\mathbf{R} &= \frac{1}{4}\frac{(-1)^s}{4!}\boldsymbol{\varepsilon}_{\mu\nu\alpha\beta}\boldsymbol{\varepsilon}^{\mu\nu\rho\sigma}\boldsymbol{\varepsilon}_{\lambda\kappa\tau\gamma}\mathbf{R}^{\alpha\beta}{}_{\rho\sigma}\theta^\lambda\wedge\theta^\kappa\wedge\theta^\tau\wedge\theta^\gamma \\ &= \frac{1}{4}\frac{(-1)^s(-1)^s 4!}{4!}\boldsymbol{\varepsilon}_{\mu\nu\alpha\beta}\delta^{[\mu}_{\lambda}\delta^{\nu}_{\kappa}\delta^{\rho}_{\tau}\delta^{\sigma]}_{\gamma}\mathbf{R}^{\alpha\beta}{}_{\rho\sigma}\theta^\lambda\wedge\theta^\kappa\wedge\theta^\tau\wedge\theta^\gamma \\ &= \frac{1}{4}\boldsymbol{\varepsilon}_{\mu\nu\alpha\beta}\mathbf{R}^{\alpha\beta}{}_{\rho\sigma}\theta^\mu\wedge\theta^\nu\wedge\theta^\rho\wedge\theta^\sigma = e\frac{1}{2}\boldsymbol{\varepsilon}_{\mu\nu\alpha\beta}\theta^\mu\wedge\theta^\nu\wedge\mathbf{R}^{\alpha\beta} \end{aligned}$$

where the last equality is obtained (when the basis is holonomic, $\theta^\mu = \mathrm{d}x^\mu$) since the curvature 2 form as $\mathbf{R}^{\alpha\beta} = \frac{1}{2}\mathbf{R}^{\alpha\beta}{}_{\rho\sigma}\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma$. Finally, using the relation that relation between the volume element $\varepsilon_{\mu\nu\alpha\beta}$ of $g_{\mu\nu} = e^I_\mu e^J_\nu\eta_{IJ}$ and the volume element ε_{IJKL} of η_{IJ}, namely : $\boldsymbol{\varepsilon}_{\mu\nu\alpha\beta} = e^I_\mu e^J_\nu e^K_\alpha e^L_\beta\varepsilon_{IJKL}$, then we write :

$$\begin{aligned} \mathcal{L}_{\mathsf{EH}} &= \frac{1}{2}\int_{\mathcal{X}} e^I_\mu e^J_\nu e^K_\alpha e^L_\beta\varepsilon_{IJKL}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathbf{R}^{\alpha\beta} = \frac{1}{2}\int_{\mathcal{X}}\varepsilon_{IJKL}e^I_\mu\mathrm{d}x^\mu\wedge e^J_\nu\mathrm{d}x^\nu\wedge e^K_\alpha e^L_\beta\mathbf{R}^{\alpha\beta} \\ &= \frac{1}{2}\int_{\mathcal{X}}\varepsilon_{IJKL}e^I\wedge e^J\wedge\mathbf{F}^{KL} \end{aligned}$$

Notice that in an tetrad frame, we have : $\varepsilon_{IJKL} = \boldsymbol{\varepsilon}_{IJKL}$ so that we also may write the integrand of the action as $1/2\boldsymbol{\varepsilon}_{IJKL}e^I\wedge e^J\wedge\mathbf{F}^{KL}$ ⌋.

The previous calculation is useful for the following one:

$$\begin{aligned} \star\mathfrak{U}_{\alpha\beta}\wedge\mathbf{R}^{\alpha\beta} &= \big[\frac{1}{2}\boldsymbol{\varepsilon}_{\alpha\beta\rho\sigma}\theta^\rho\wedge\theta^\sigma\big]\wedge\big[\frac{1}{2}\mathbf{R}^{\alpha\beta}{}_{\mu\nu}\theta^\mu\wedge\theta^\nu\big] = \frac{1}{2}\mathbf{R}^{\alpha\beta}{}_{\mu\nu}\big[\frac{1}{2}\boldsymbol{\varepsilon}_{\alpha\beta\rho\sigma}\theta^\rho\wedge\theta^\sigma\wedge\theta^\mu\wedge\theta^\nu\big] \\ &= \frac{1}{2}\mathbf{R}^{\alpha\beta}{}_{\mu\nu}\big[\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha\big]\mathrm{d}\mathfrak{y} = \mathbf{R}\mathrm{d}\mathfrak{y} = \star\mathbf{R} \quad ⌋ \end{aligned}$$

1.2.6 Internal (☆) and external (⋆) Hodge operators

In this section we introduce two operators: the *internal* ☆ and the *external* ⋆ Hodge operator in the context of vector (Lie algebra)-valued p-form. Let $\lambda \in \Omega^p(\mathcal{X},\mathcal{V}) = \Omega^p(\mathcal{X})\otimes\mathcal{V}$ a $\mathcal{V}$-valued p-form. Naturally we construct the space $\Omega^n(\mathcal{X})\otimes\Lambda^n(\mathcal{V})$ and we consider the case where $\dim(\mathcal{X}) = \dim(\mathcal{V}) = n$. Then, let $0 \le p, q \le n$ we describe a $\Lambda^q(\mathcal{V})$-valued p-form on which we apply an Hodge operator either on *internal* indices or on *space-time* indices.

$$☆ : \left\{ \begin{array}{ccc} \Omega^p(\mathcal{M})\otimes\Lambda^q\mathcal{V} & \to & \Omega^p(\mathcal{M})\otimes\Lambda^{n-q}\mathcal{V} \\ \boldsymbol{\varphi} & \mapsto & ☆\boldsymbol{\varphi} \end{array}\right. \tag{1.26}$$

and

$$\star : \left\{ \begin{array}{ccc} \Omega^p(\mathcal{M})\otimes\Lambda^q\mathcal{V} & \to & \Omega^{n-p}(\mathcal{M})\otimes\Lambda^q\mathcal{V} \\ \boldsymbol{\varphi} & \mapsto & \star\boldsymbol{\varphi} \end{array}\right. \tag{1.27}$$

Definition 1.2. *Hodge duality for vector-valued p-form $\boldsymbol{\varphi} \in \Omega^p(\mathcal{X},\mathcal{V})$. We consider $\{\theta_\mu\}$ a basis of $\mathcal{V}$ and $\mathrm{d}x^\mu$ the holonomic basis of $T^\star\mathcal{X}$. Therefore, the basis of the space $\Lambda^p(\mathcal{X})\otimes\Lambda^q(\mathcal{V})$ is $\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_n}\otimes\theta_{I_1}\wedge\cdots\wedge\theta_{I_n}$ Then: $\boldsymbol{\varphi} = \frac{1}{p!q!}\Big[\boldsymbol{\varphi}\Big]^{I_1\cdots I_q}_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}\otimes\theta_{I_1}\wedge\cdots\wedge\theta_{I_q}$ and*

$$\begin{aligned}\star\boldsymbol{\varphi} &= \frac{1}{(n-p)!}\Big[\star\boldsymbol{\varphi}\Big]^{I_1\cdots I_q}_{\nu_1\cdots\nu_{n-p}}\mathrm{d}x^{\nu_1}\wedge\cdots\wedge\mathrm{d}x^{\nu_{n-p}}\otimes\theta_{I_1}\wedge\cdots\wedge\theta_{I_q}\\ \star\boldsymbol{\varphi} &= \frac{1}{(n-q)!}\Big[\star\boldsymbol{\varphi}\Big]^{J_1\cdots J_{n-q}}_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}\otimes\theta_{J_1}\wedge\cdots\wedge\theta_{J_{n-q}}\end{aligned} \tag{1.28}$$

with

$$\begin{aligned}\Big[\star\boldsymbol{\varphi}\Big]^{I_1\cdots I_q}_{\nu_1\cdots\nu_{n-p}} &= \star\boldsymbol{\varphi}_{\nu_1\cdots\nu_{n-p}} = \frac{1}{p!}\varepsilon^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_{n-p}}\boldsymbol{\varphi}_{\mu_1\cdots\mu_p}\\ \Big[\star\boldsymbol{\varphi}\Big]^{J_1\cdots J_{n-q}}_{\mu_1\cdots\mu_p} &= \star\boldsymbol{\varphi}^{J_1\cdots J_{n-q}} = \frac{1}{q!}\varepsilon^{J_1\cdots J_{n-q}}{}_{I_1\cdots I_q}\boldsymbol{\varphi}^{I_1\cdots I_q}\end{aligned} \tag{1.29}$$

1.3 Self Duality

1.3.1 Internal Hodge operator in 4D manifolds

Self Duality is widely considered as a main construction in mathematical physics. If we consider a $\mathfrak{so}(1,3)$-valued two forms. We denote by $\boldsymbol{\Delta}_{IJ}$ the generators of the $\mathfrak{so}(1,3)$ Lie algebra. We also denote $\boldsymbol{\mathcal{V}}^q = \boldsymbol{\mathcal{V}}^q_\wedge = \bigwedge_{1\leq j\leq q}\mathcal{V}$. We focus here on the case where $\dim(\mathcal{X}) = \dim(\mathcal{V}) = n$ and so we consider $\boldsymbol{\mathcal{V}}^q$-valued p-forms with $0\leq q,p\leq n$.

$\boldsymbol{\mathcal{V}}^q$-valued p-**forms and theirs $\star$-dual** For a $\boldsymbol{\mathcal{V}}^q$-valued 0-form, $\boldsymbol{\eta}_0\in\Omega^0(\mathcal{X},\boldsymbol{\mathcal{V}}^q) = \Omega^0(\mathcal{X})\otimes\boldsymbol{\mathcal{V}}^q$

$$\left|\begin{aligned}\boldsymbol{\eta}_0 &= \frac{1}{q!}\big[\boldsymbol{\eta}_0\big]^{I_1\cdots I_q}\otimes\mathfrak{e}_{I_1}\wedge\cdots\wedge\mathfrak{e}_{I_q}\\ \star\boldsymbol{\eta}_0 &= \frac{1}{4!}(\star\boldsymbol{\eta}_0)^{IJ}_{\mu\nu\rho\sigma}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma\otimes\mathfrak{e}_{I_1}\wedge\cdots\wedge\mathfrak{e}_{I_q}\text{ with }(\star\boldsymbol{\eta}_0)^{I_1\cdots I_q}_{\mu\nu\rho\sigma} = \varepsilon_{\mu\nu\rho\sigma}(\boldsymbol{\eta}_0)^{I_1\cdots I_q}\end{aligned}\right. \tag{1.30}$$

For a $\boldsymbol{\mathcal{V}}^q$-valued Lie algebra 1-form, $\boldsymbol{\eta}_1\in\Omega^1(\mathcal{X},\boldsymbol{\mathcal{V}}^q) = \Omega^1(\mathcal{X})\otimes\boldsymbol{\mathcal{V}}^q$ and $\boldsymbol{\eta}_2\in\Omega^2(\mathcal{X},\boldsymbol{\mathcal{V}}^q) = \Omega^2(\mathcal{X})\otimes\boldsymbol{\mathcal{V}}^q$

$$\left|\begin{aligned}\boldsymbol{\eta}_1 &= \frac{1}{q!}\big[\boldsymbol{\eta}\big]^{I_1\cdots I_q}_{\mu}\mathrm{d}x^\mu\otimes\mathfrak{e}_{I_1}\wedge\cdots\wedge\mathfrak{e}_{I_q}\\ \star\boldsymbol{\eta}_1 &= \frac{1}{3!}(\star\boldsymbol{\eta}_1)^{IJ}_{\nu\rho\sigma}\mathrm{d}x^\nu\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^\rho\otimes\mathfrak{e}_{I_1}\wedge\cdots\wedge\mathfrak{e}_{I_q}\text{ with }(\star\boldsymbol{\eta}_1)^{I_1\cdots I_q}_{\nu\rho\sigma} = \varepsilon^{\mu}{}_{\nu\rho\sigma}(\boldsymbol{\eta}_1)^{I_1\cdots I_q}_{\mu}\end{aligned}\right. \tag{1.31}$$

$$\left|\begin{aligned}\boldsymbol{\eta}_2 &= \frac{1}{2!q!}\big[\boldsymbol{\eta}_2\big]^{I_1\cdots I_q}_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\otimes\mathfrak{e}_{I_1}\wedge\cdots\wedge\mathfrak{e}_{I_q}\\ \star\boldsymbol{\eta}_2 &= \frac{1}{2!}(\star\boldsymbol{\eta}_2)^{I_1\cdots I_q}_{\rho\sigma}\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma\otimes\mathfrak{e}_{I_1}\wedge\cdots\wedge\mathfrak{e}_{I_q}\text{ with }(\star\boldsymbol{\eta}_2)^{I_1\cdots I_q}_{\rho\sigma} = \frac{1}{2}\varepsilon^{\mu\nu}{}_{\rho\sigma}(\boldsymbol{\eta}_2)^{I_1\cdots I_q}_{\mu\nu}\end{aligned}\right. \tag{1.32}$$

Now we consider the $\boldsymbol{\eta}_3\in\Omega^3(\mathcal{X},\boldsymbol{\mathcal{V}}^q) = \Omega^3(\mathcal{X})\otimes\boldsymbol{\mathcal{V}}^q$ and $\boldsymbol{\eta}_4\in\Omega^4(\mathcal{X},\boldsymbol{\mathcal{V}}^q) = \Omega^4(\mathcal{X})\otimes\boldsymbol{\mathcal{V}}^q$

$$\left|\begin{aligned}\boldsymbol{\eta}_3 &= \frac{1}{3!q!}\big[\boldsymbol{\eta}_3\big]^{I_1\cdots I_q}_{\mu\nu\rho}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\otimes\mathfrak{e}_{I_1}\wedge\cdots\wedge\mathfrak{e}_{I_q}\\ \star\boldsymbol{\eta}_3 &= (\star\boldsymbol{\eta}_3)^{I_1\cdots I_q}_{\sigma}\mathrm{d}x^\sigma\otimes\mathfrak{e}_{I_1}\wedge\cdots\wedge\mathfrak{e}_{I_q}\text{ with }(\star\boldsymbol{\eta}_3)^{IJ}_{\sigma} = \frac{1}{3!}\varepsilon^{\mu\nu\rho}{}_{\sigma}\boldsymbol{\eta}_3{}^{I_1\cdots I_q}_{\mu\nu\rho}\end{aligned}\right. \tag{1.33}$$

$$\begin{vmatrix} \boldsymbol{\eta}_4 & = & \frac{1}{4!q!}\left[\boldsymbol{\eta}_4\right]^{I_1\cdots I_q}_{\mu\nu\rho\sigma}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma\otimes\mathfrak{e}_{I_1}\wedge\cdots\wedge\mathfrak{e}_{I_q} \\ \star\boldsymbol{\eta}_4 & = & (\star\boldsymbol{\eta}_4)^{I_1\cdots I_q}\otimes\mathfrak{e}_{I_1}\wedge\cdots\wedge\mathfrak{e}_{I_q} \text{ with } \quad (\star\boldsymbol{\eta}_4)^{I_1\cdots I_q} = \frac{1}{4!}\varepsilon^{\mu\nu\rho\sigma}(\boldsymbol{\eta}_4)^{I_1\cdots I_q}_{\mu\nu\rho\sigma} \end{vmatrix} \tag{1.34}$$

$\boldsymbol{\mathcal{V}}^q$-valued p-**forms and theirs** $\star$**-dual**. For a $\boldsymbol{\mathcal{V}}^0$-valued p-form, $\boldsymbol{\eta}\in\Omega^p(\mathcal{X},\boldsymbol{\mathcal{V}}^q)=\Omega^p(\mathcal{X})\otimes\boldsymbol{\mathcal{V}}^0$

$$\begin{vmatrix} \boldsymbol{\eta} & = & \frac{1}{p!}\left[\boldsymbol{\eta}\right]_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\mathrm{d}x^{\mu_p} \\ \star\boldsymbol{\eta} & = & \frac{1}{4!}(\star\boldsymbol{\eta})^{IJKL}_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}\otimes\mathfrak{e}_I\wedge\mathfrak{e}_J\wedge\mathfrak{e}_K\wedge\mathfrak{e}_L \text{ with } (\star\boldsymbol{\eta})^{IJKL}_{\mu_1\cdots\mu_p} = \varepsilon^{IJKL}(\boldsymbol{\eta})_{\mu_1\cdots\mu_p} \end{vmatrix} \tag{1.35}$$

For a $\boldsymbol{\mathcal{V}}^1$-valued Lie algebra p-form, $\boldsymbol{\eta}\in\Omega^p(\mathcal{X},\boldsymbol{\mathcal{V}}^1)=\Omega^p(\mathcal{X})\otimes\boldsymbol{\mathcal{V}}^1$ and the $\boldsymbol{\eta}\in\Omega^p(\mathcal{X},\boldsymbol{\mathcal{V}}^2)=\Omega^p(\mathcal{X})\otimes\boldsymbol{\mathcal{V}}^2$

$$\begin{vmatrix} \boldsymbol{\eta} & = & \frac{1}{p!}\left[\boldsymbol{\eta}\right]^I_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\mathrm{d}x^{\mu_p}\otimes\mathfrak{e}_I \\ \star\boldsymbol{\eta} & = & \frac{1}{3!}(\star\boldsymbol{\eta})^{JKL}_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}\otimes\mathfrak{e}_J\wedge\mathfrak{e}_K\wedge\mathfrak{e}_L \text{ with } (\star\boldsymbol{\eta})^{JKL}_{\mu_1\cdots\mu_p} = \varepsilon^{JKL}{}_I(\boldsymbol{\eta})^I_{\mu_1\cdots\mu_p} \end{vmatrix} \tag{1.36}$$

$$\begin{vmatrix} \boldsymbol{\eta} & = & \frac{1}{p!2!}\left[\boldsymbol{\eta}\right]^{IJ}_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}\otimes\mathfrak{e}_I\wedge\mathfrak{e}_J \\ \star\boldsymbol{\eta} & = & \frac{1}{2!}(\star\boldsymbol{\eta})^{KL}_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}\otimes\mathfrak{e}_J\wedge\mathfrak{e}_K \text{ with } (\star\boldsymbol{\eta})^{KL}_{\mu_1\cdots\mu_p} = \frac{1}{2}\varepsilon^{KL}{}_{IJ}(\boldsymbol{\eta})^{KL}_{\mu\nu} \end{vmatrix} \tag{1.37}$$

Now we consider the $\boldsymbol{\eta}_3\in\Omega^3(\mathcal{X},\boldsymbol{\mathcal{V}}^q)=\Omega^3(\mathcal{X})\otimes\boldsymbol{\mathcal{V}}^q$ and $\boldsymbol{\eta}\in\Omega^p(\mathcal{X},\boldsymbol{\mathcal{V}}^4)=\Omega^p(\mathcal{X})\otimes\boldsymbol{\mathcal{V}}^4$

$$\begin{vmatrix} \boldsymbol{\eta} & = & \frac{1}{p!3!}\left[\boldsymbol{\eta}_3\right]^{IJK}_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}\otimes\mathfrak{e}_I\wedge\mathfrak{e}_J\wedge\mathfrak{e}_K \\ \star\boldsymbol{\eta} & = & (\star\boldsymbol{\eta})^L_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}\otimes\mathfrak{e}_L \text{ with } (\star\boldsymbol{\eta})^L_{\mu_1\cdots\mu_p} = \frac{1}{3!}\varepsilon^L{}_{IJK}\boldsymbol{\eta}^{IJK}_{\mu_1\cdots\mu_p} \end{vmatrix} \tag{1.38}$$

$$\begin{vmatrix} \boldsymbol{\eta} & = & \frac{1}{p!4!}\left[\boldsymbol{\eta}\right]^{IJKL}_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}\otimes\mathfrak{e}_I\wedge\mathfrak{e}_J\wedge\mathfrak{e}_K\wedge\mathfrak{e}_L \\ \star\boldsymbol{\eta} & = & (\star\boldsymbol{\eta})_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p} \text{ with } \quad (\star\boldsymbol{\eta})_{\mu_1\cdots\mu_p} = \frac{1}{4!}\varepsilon_{IJKL}(\boldsymbol{\eta})^{IJKL}_{\mu_1\cdots\mu_p} \end{vmatrix} \tag{1.39}$$

In fact if we want to explore the internal Hodge dual it is good to describe the $\mathfrak{so}(3,1)$-Lie algebra in term of bi-vectors. We will enter into details of this isomorphism in (1.5.5) below. $\boldsymbol{\Delta}_{IJ}\cong\mathfrak{e}_I\wedge\mathfrak{e}_J$. Notice that in the following we mainly focus on the following cases: $\boldsymbol{\eta}\in\Omega^2(\mathcal{X},\Lambda^2\mathcal{V})=\Omega^2(\mathcal{X})\otimes\Lambda^2\mathcal{V}$ and $\boldsymbol{\rho}\in\Omega^2(\mathcal{X},\mathfrak{so}(3,1))=\Omega^2(\mathcal{X})\otimes\mathfrak{so}(3,1)$

$$\begin{vmatrix} \boldsymbol{\eta} & = & \frac{1}{2!2!}\left[\boldsymbol{\eta}\right]^{IJ}_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\otimes\mathfrak{e}_I\wedge\mathfrak{e}_J \\ \boldsymbol{\rho} & = & \frac{1}{2!}\left[\boldsymbol{\rho}\right]^{IJ}_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\otimes\boldsymbol{\Delta}_{IJ} \end{vmatrix} \tag{1.40}$$

1.3.2 Hodge operator on 2-form in Riemannian and Lorentzian 4D manifold

$(\mathcal{X}, g)$ is a Riemanian manifold We consider a form $\boldsymbol{\eta}$ so that:

$$\star\star\boldsymbol{\eta} = (-1)^{k(n+1)}\boldsymbol{\eta} \tag{1.41}$$

with $n = 4$, $k = 2$ we have: $\star\star\boldsymbol{\eta} = (-1)^{2(4+1)}\boldsymbol{\eta} = \boldsymbol{\eta}$. In the Riemanian case: $\star\star = \mathbf{Id}$ and $\smallstar\smallstar = \mathbf{Id}$.

$$\begin{array}{llll} \text{Left-handed} & \star\big[\boldsymbol{\eta}^{\blacktriangleleft}\big] = \boldsymbol{\eta}^{\blacktriangleleft} & \text{Self-dual} & \smallstar\big[\boldsymbol{\eta}^{\triangleleft}\big] = \boldsymbol{\eta}^{\triangleleft} \\ \text{Right-handed} & \star\big[\boldsymbol{\eta}^{\blacktriangleright}\big] = -\boldsymbol{\eta}^{\blacktriangleright} & \text{Anti-self-dual} & \smallstar\big[\boldsymbol{\eta}^{\triangleright}\big] = -\boldsymbol{\eta}^{\triangleright} \end{array} \tag{1.42}$$

$(\mathcal{X}, g)$ **is a Lorentzian manifold of signature** $(3,1)$. In this case, we have:

$$\star\star\boldsymbol{\eta} = (-1)^{1+k(n-k)}\boldsymbol{\eta} = (-1)^{1+2(4+1)}\boldsymbol{\eta} = -\boldsymbol{\eta}$$

$$\star\star = -\mathbf{Id} \qquad \text{and} \qquad \smallstar\smallstar = -\mathbf{Id}$$

$$\begin{array}{llll} \text{Left-handed} & \star\big[\boldsymbol{\eta}^{\blacktriangleleft}\big] = i\boldsymbol{\eta}^{\blacktriangleleft} & \text{Self-dual} & \smallstar\big[\boldsymbol{\eta}^{\triangleleft}\big] = i\boldsymbol{\eta}^{\triangleleft} \\ \text{Right-handed} & \star\big[\boldsymbol{\eta}^{\blacktriangleright}\big] = -i\boldsymbol{\eta}^{\blacktriangleright} & \text{Anti-self-dual} & \smallstar\big[\boldsymbol{\eta}^{\triangleright}\big] = -i\boldsymbol{\eta}^{\triangleright} \end{array} \tag{1.43}$$

so that for the curvature 2-form, considered on a Lorentzian manifold, with $(\mathcal{X}, g)$ of signature, $(3,1)$

$$\begin{array}{llll} \text{Left-handed} & \star\big[\mathbf{F}^{\blacktriangleleft}\big] = i\mathbf{F}^{\blacktriangleleft} & \text{Self-dual} & \smallstar\big[\mathbf{F}^{\triangleleft}\big] = i\mathbf{F}^{\triangleleft} \\ \text{Right-handed} & \star\big[\mathbf{F}^{\blacktriangleright}\big] = -i\mathbf{F}^{\blacktriangleright} & \text{Anti-self-dual} & \smallstar\big[\mathbf{F}^{\triangleright}\big] = -i\mathbf{F}^{\triangleright} \end{array} \tag{1.44}$$

We consider the following object: left-handed and right-handed and self-dual and antiself-dual.

$$\begin{array}{llll} \text{Left-handed} & \boldsymbol{\eta}^{\blacktriangleleft} = (1/2)\big[\boldsymbol{\eta} - i\star(\boldsymbol{\eta})\big] & \text{Self-dual} & \boldsymbol{\eta}^{\triangleleft} = (1/2)\big[\boldsymbol{\eta} - i\smallstar(\boldsymbol{\eta})\big] \\ \text{Right-handed} & \boldsymbol{\eta}^{\blacktriangleright} = (1/2)\big[\boldsymbol{\eta} + i\star(\boldsymbol{\eta})\big] & \text{Anti-self-dual} & \boldsymbol{\eta}^{\triangleright} = (1/2)\big[\boldsymbol{\eta} + i\smallstar(\boldsymbol{\eta})\big] \end{array} \tag{1.45}$$

We write in components

$$\begin{array}{ll} \text{Left-handed} & \big[\boldsymbol{\eta}^{\blacktriangleleft}\big]^{IJ} = (1/2)\big[\boldsymbol{\eta}^{IJ} - i\star(\boldsymbol{\eta}^{IJ})\big] \\ \text{Right-handed} & \big[\boldsymbol{\eta}^{\blacktriangleright}\big]^{IJ} = (1/2)\big[\boldsymbol{\eta}^{IJ} + i\star(\boldsymbol{\eta}^{IJ})\big] \\ \text{Self-dual} & \big[\boldsymbol{\eta}^{\triangleleft}\big]^{IJ} = (1/2)\big[\boldsymbol{\eta}^{IJ} - i\smallstar(\boldsymbol{\eta}^{IJ})\big] \\ \text{Anti-self-dual} & \big[\boldsymbol{\eta}^{\triangleright}\big]^{IJ} = (1/2)\big[\boldsymbol{\eta}^{IJ} + i\smallstar(\boldsymbol{\eta}^{IJ})\big] \end{array} \tag{1.46}$$

1.3.3 Self Dual Projector

The projection operators are defined:

$$\mathfrak{P}^{\triangleleft} = \underbrace{\frac{1}{2}\big[1 - i\smallstar\big]}_{\text{Self-Dual Projector}} \qquad \mathfrak{P}^{\triangleright} = \underbrace{\frac{1}{2}\big[1 + i\smallstar\big]}_{\text{Anti-Self-Dual Projector}}$$

Notice that:

$$(\mathfrak{P}^{\triangleleft}\boldsymbol{\eta})^{IJ} = (\frac{1}{2}\big[1 - i\smallstar\big]\boldsymbol{\eta})^{IJ} = \frac{1}{2}\big[\delta^I_K\delta^J_L - \frac{i}{2}\varepsilon^{IJ}{}_{KL}\big]\boldsymbol{\eta}^{KL} = \frac{1}{2}\big[\boldsymbol{\eta} - \frac{i}{2}\varepsilon^{IJ}{}_{KL}\boldsymbol{\eta}^{KL}_{\mu\nu}\big] = (\boldsymbol{\eta}^{\triangleleft})^{IJ}$$

so that: $\mathfrak{p}^{\triangleleft}\boldsymbol{\eta} = \boldsymbol{\eta}^{\triangleleft}$ and $\mathfrak{p}^{\triangleright}\boldsymbol{\eta} = \boldsymbol{\eta}^{\triangleright}$. If we would have work with the external star operator, we find:

$$\mathfrak{p}^{\blacktriangleleft} = \underbrace{\frac{1}{2}\big[1 + i\star\big]}_{\text{Left-handed Projector}} \qquad \mathfrak{p}^{\blacktriangleright} = \underbrace{\frac{1}{2}\big[1 - i\star\big]}_{\text{Right-Handed Projector}}$$

We consider the generators of the Lorentz group $\mathsf{SO}(1,3)$, which are denoted by $\boldsymbol{\Delta}^{\mathsf{SO}(1,3)}_{IJ} = \boldsymbol{\Delta}_{IJ}$ so that the commutation relations are written:

$$[\boldsymbol{\Delta}_{IJ}, \boldsymbol{\Delta}_{KL}] = \eta_{IL}\boldsymbol{\Delta}_{JK} + \eta_{JK}\boldsymbol{\Delta}_{IL} - \eta_{IK}\boldsymbol{\Delta}_{JL} - \eta_{JL}\boldsymbol{\Delta}_{IK}$$

Recall that[17]:

$$[\boldsymbol{\eta}, \boldsymbol{\rho}]^{IJ} = \boldsymbol{\eta}^{I}{}_{K}\boldsymbol{\rho}^{KJ} - \boldsymbol{\rho}^{I}{}_{K}\boldsymbol{\eta}^{KJ}$$

So that: $[\boldsymbol{\eta}, \boldsymbol{\rho}] = \frac{1}{2!}\frac{1}{2!}2\Big[\boldsymbol{\eta}^{I}{}_{K}\boldsymbol{\rho}^{KJ} - \boldsymbol{\rho}^{I}{}_{K}\boldsymbol{\eta}^{KJ}\Big] \otimes \boldsymbol{\Delta}_{IJ}$. Since $\star[\boldsymbol{\eta}, \star\boldsymbol{\rho}] = \star[\boldsymbol{\eta}, \star\boldsymbol{\rho}]^{IJ} \otimes \boldsymbol{\Delta}_{IJ} \in \Omega^4(\mathcal{X}) \otimes \mathfrak{so}(1,3)$ we obtain[18]:

$$\star[\boldsymbol{\eta}, \star\boldsymbol{\rho}]^{IJ} \quad = \quad \star\Big[\boldsymbol{\eta}^{I}{}_{K}\star(\boldsymbol{\rho}^{KJ}) - \star(\boldsymbol{\rho}^{I}{}_{K})\boldsymbol{\eta}^{KJ}\Big]$$

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[17] ⌈ **Proof** For $\boldsymbol{\eta}, \boldsymbol{\rho} \in \Omega^2(\mathcal{X}) \otimes \mathfrak{so}(3,1)$ we have: $[\boldsymbol{\eta}, \boldsymbol{\rho}] \in \Omega^4(\mathcal{X}) \otimes \mathfrak{so}(3,1)$ so that:

$$\begin{aligned}
[\boldsymbol{\eta}, \boldsymbol{\rho}] &= \boldsymbol{\eta}^{IJ} \wedge \boldsymbol{\rho}^{KL} \otimes [\boldsymbol{\Delta}_{IJ}, \boldsymbol{\Delta}_{KL}] \\
&= \boldsymbol{\eta}^{IJ} \wedge \boldsymbol{\rho}^{KL} \otimes \eta_{IL}\boldsymbol{\Delta}_{JK} + \eta_{JK}\boldsymbol{\Delta}_{IL} - \eta_{IK}\boldsymbol{\Delta}_{JL} - \eta_{JL}\boldsymbol{\Delta}_{IK} \\
&= \boldsymbol{\eta}^{IJ}\boldsymbol{\rho}^{KL}\mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\rho} \wedge \mathrm{d}x^{\sigma} \otimes \eta_{IL}\boldsymbol{\Delta}_{JK} + \eta_{JK}\boldsymbol{\Delta}_{IL} - \eta_{IK}\boldsymbol{\Delta}_{JL} - \eta_{JL}\boldsymbol{\Delta}_{IK} \\
&= \underbrace{\boldsymbol{\eta}^{IJ}\boldsymbol{\rho}^{KL} \otimes \eta_{IL}\boldsymbol{\Delta}_{JK}}_{\boldsymbol{\kappa}_1} \underbrace{+\boldsymbol{\eta}^{IJ}\boldsymbol{\rho}^{KL} \otimes \eta_{JK}\boldsymbol{\Delta}_{IL}}_{\boldsymbol{\kappa}_2} \underbrace{-\boldsymbol{\eta}^{IJ}\boldsymbol{\rho}^{KL} \otimes \eta_{IK}\boldsymbol{\Delta}_{JL}}_{\boldsymbol{\kappa}_3} \underbrace{-\boldsymbol{\eta}^{IJ}\boldsymbol{\rho}^{KL} \otimes \eta_{JL}\boldsymbol{\Delta}_{IK}}_{\boldsymbol{\kappa}_4}
\end{aligned}$$

with

$$\begin{aligned}
\boldsymbol{\kappa}_3 &= -\boldsymbol{\eta}^{IJ}\boldsymbol{\rho}^{KL} \otimes \eta_{IK}\boldsymbol{\Delta}_{JL} = -(\eta_{IK}\boldsymbol{\eta}^{IJ})\boldsymbol{\rho}^{KL} \otimes \boldsymbol{\Delta}_{JL} = -\boldsymbol{\eta}_{K}{}^{J}\boldsymbol{\rho}^{KL} \otimes \boldsymbol{\Delta}_{JL} = -\boldsymbol{\eta}_{K}{}^{I}\boldsymbol{\rho}^{KJ} \otimes \boldsymbol{\Delta}_{IJ} \\
\boldsymbol{\kappa}_1 &= \boldsymbol{\eta}^{IJ}\boldsymbol{\rho}^{KL} \otimes \eta_{IL}\boldsymbol{\Delta}_{JK} = (\eta_{IL}\boldsymbol{\eta}^{IJ})\boldsymbol{\rho}^{KL} \otimes \boldsymbol{\Delta}_{JK} = \boldsymbol{\eta}_{L}{}^{J}\boldsymbol{\rho}^{KL} \otimes \boldsymbol{\Delta}_{JK} = \boldsymbol{\eta}_{L}{}^{I}\boldsymbol{\rho}^{JL} \otimes \boldsymbol{\Delta}_{IJ} \\
\boldsymbol{\kappa}_2 &= \boldsymbol{\eta}^{IJ}\boldsymbol{\rho}^{KL} \otimes \eta_{JK}\boldsymbol{\Delta}_{IL} = \boldsymbol{\eta}^{IJ}(\eta_{JK}\boldsymbol{\rho}^{KL}) \otimes \boldsymbol{\Delta}_{IL} = -\boldsymbol{\eta}^{IJ}(\eta_{KJ}\boldsymbol{\rho}^{KL}) \otimes \boldsymbol{\Delta}_{IL} = -\boldsymbol{\eta}^{IL}\boldsymbol{\rho}_{L}{}^{J} \otimes \boldsymbol{\Delta}_{IJ} \\
\boldsymbol{\kappa}_4 &= -\boldsymbol{\eta}^{IJ}\boldsymbol{\rho}^{KL} \otimes \eta_{JL}\boldsymbol{\Delta}_{IK} = -\boldsymbol{\eta}^{IJ}(\eta_{JL}\boldsymbol{\rho}^{KL}) \otimes \boldsymbol{\Delta}_{IK} = -\boldsymbol{\eta}^{IJ}\boldsymbol{\rho}^{K}{}_{J} \otimes \boldsymbol{\Delta}_{IK} \\
&= -\boldsymbol{\eta}^{IJ}\boldsymbol{\rho}^{KL} \otimes \eta_{JL}\boldsymbol{\Delta}_{IK} = -\boldsymbol{\eta}^{IJ}(\eta_{JL}\boldsymbol{\rho}^{KL}) \otimes \boldsymbol{\Delta}_{IK} = -\boldsymbol{\eta}^{IK}\boldsymbol{\rho}^{J}{}_{K} \otimes \boldsymbol{\Delta}_{IJ}
\end{aligned}$$

So that :

$$\begin{aligned}
[\boldsymbol{\eta}, \boldsymbol{\rho}] &= \big[\boldsymbol{\eta}_{L}{}^{I}\boldsymbol{\rho}^{JL} - \boldsymbol{\eta}^{IL}\boldsymbol{\rho}_{L}{}^{J} - \boldsymbol{\eta}_{K}{}^{I}\boldsymbol{\rho}^{KJ} - \boldsymbol{\eta}^{IK}\boldsymbol{\rho}^{J}{}_{K}\big] \otimes \boldsymbol{\Delta}_{IJ} = \big[\boldsymbol{\eta}^{I}{}_{K}\boldsymbol{\rho}^{KJ} + \boldsymbol{\eta}_{K}{}^{I}\boldsymbol{\rho}^{JK} - \boldsymbol{\eta}^{IK}\boldsymbol{\rho}_{K}{}^{J} - \boldsymbol{\eta}^{IK}\boldsymbol{\rho}^{J}{}_{K}\big] \otimes \boldsymbol{\Delta}_{IJ} \\
&= \big[\boldsymbol{\eta}^{I}{}_{K}\boldsymbol{\rho}^{KJ} - \boldsymbol{\eta}_{K}{}^{I}\boldsymbol{\rho}^{KJ} - (\boldsymbol{\rho}_{K}{}^{J} + \boldsymbol{\rho}^{J}{}_{K})\boldsymbol{\eta}^{IK}\big] \otimes \boldsymbol{\Delta}_{IJ} = \big[\boldsymbol{\eta}^{I}{}_{K}\boldsymbol{\rho}^{KJ} + \boldsymbol{\eta}^{I}{}_{K}\boldsymbol{\rho}^{KJ} - 2\boldsymbol{\rho}^{J}{}_{K}\boldsymbol{\eta}^{IK}\big] \otimes \boldsymbol{\Delta}_{IJ} \\
&= 2\big[\boldsymbol{\eta}^{I}{}_{K}\boldsymbol{\rho}^{KJ} - \boldsymbol{\rho}^{J}{}_{K}\boldsymbol{\eta}^{IK}\big] \otimes \boldsymbol{\Delta}_{IJ} = 2\big[\boldsymbol{\eta}^{I}{}_{K}\boldsymbol{\rho}^{KJ}\big] \otimes \boldsymbol{\Delta}_{IJ} + 2\big[\boldsymbol{\rho}^{J}{}_{K}\boldsymbol{\eta}^{KI}\big] \otimes \boldsymbol{\Delta}_{IJ} \\
&= 2\big[\boldsymbol{\eta}^{I}{}_{K}\boldsymbol{\rho}^{KJ}\big] \otimes \boldsymbol{\Delta}_{IJ} + 2\big[\boldsymbol{\rho}^{I}{}_{K}\boldsymbol{\eta}^{KJ}\big] \otimes \boldsymbol{\Delta}_{JI}
\end{aligned}$$

But since $\boldsymbol{\Delta}_{IJ} = -\boldsymbol{\Delta}_{JI}$ we obtain: $[\boldsymbol{\eta}, \boldsymbol{\rho}] = 2\Big[\boldsymbol{\eta}^{I}{}_{K}\boldsymbol{\rho}^{KJ} - \boldsymbol{\rho}^{I}{}_{K}\boldsymbol{\eta}^{KJ}\Big] \otimes \boldsymbol{\Delta}_{IJ}$ ⌋

[18] ⌈ **Proof** Notice that $\star\boldsymbol{\rho}$ viewed as a $\mathfrak{so}(1,3)$-Lie algebra valued 2-form which is written in component: $\star\boldsymbol{\rho}^{IJ} = \frac{1}{2}\varepsilon^{IJ}{}_{KL}\boldsymbol{\rho}^{KL}$ so that:

$$\star\boldsymbol{\rho}^{I}{}_{K} = \star(\boldsymbol{\rho}^{IJ}\eta_{JK}) = \frac{1}{2}\eta_{NK}\varepsilon^{IN}{}_{ML}\boldsymbol{\rho}^{ML} \qquad \text{and} \qquad \star\boldsymbol{\rho}^{KJ} = \frac{1}{2}\varepsilon^{KJ}{}_{QR}\boldsymbol{\rho}^{QR} \tag{1.47}$$

so that: $\star[\boldsymbol{\eta}, \star\boldsymbol{\rho}]^{IJ} = \frac{1}{2}\varepsilon^{IJ}{}_{MN}[\boldsymbol{\eta}, \star\boldsymbol{\rho}]^{MN}$. But notice that; $[\boldsymbol{\eta}, \star\boldsymbol{\rho}]^{MN} = \Big[\boldsymbol{\eta}^{M}{}_{K}\star(\boldsymbol{\rho}^{KN}) - \star(\boldsymbol{\rho}^{M}{}_{K})\boldsymbol{\eta}^{KN}\Big]$ So that:

$$\star[\boldsymbol{\eta}, \star\boldsymbol{\rho}]^{MN} = \frac{1}{2}\varepsilon^{IJ}{}_{MN}\Big[\boldsymbol{\eta}^{M}{}_{K}\star(\boldsymbol{\rho}^{KN}) - \star(\boldsymbol{\rho}^{M}{}_{K})\boldsymbol{\eta}^{KN}\Big]$$

due to (1.47), we obtain:

$$\star(\boldsymbol{\rho}^{KN}) = \frac{1}{2}\varepsilon^{KN}{}_{QR}\boldsymbol{\rho}^{QR} \qquad \text{and} \qquad \star(\boldsymbol{\rho}^{M}{}_{K}) = \star(\boldsymbol{\rho}^{MS}\eta_{SK}) = \frac{1}{2}\eta_{SK}\varepsilon^{MS}{}_{QR}\boldsymbol{\rho}^{QR}$$

Recall that for $\boldsymbol{\eta},\boldsymbol{\rho} \in \Omega^2(\mathcal{X},\mathfrak{so}(3,1)) = \Omega^2(\mathcal{X})\otimes\mathfrak{so}(3,1)$, we derive the object $[\boldsymbol{\eta},\boldsymbol{\rho}] \in \Omega^4(\mathcal{X})\otimes\mathfrak{so}(3,1)$. Therefore, we equivalently write in matrix notations:

$$[\boldsymbol{\eta},\boldsymbol{\rho}] = \boldsymbol{\eta}\wedge\boldsymbol{\rho} - \boldsymbol{\rho}\wedge\boldsymbol{\eta}$$

We have the following properties :

$$\begin{array}{rcl} \star[\boldsymbol{\eta},\star\boldsymbol{\rho}] &=& -[\boldsymbol{\eta},\boldsymbol{\rho}] = \star[\star\boldsymbol{\eta},\boldsymbol{\rho}] \\ \star[\boldsymbol{\eta},\boldsymbol{\rho}] &=& [\star\boldsymbol{\eta},\boldsymbol{\rho}] = [\boldsymbol{\eta},\star\boldsymbol{\rho}] \\ [\star\boldsymbol{\eta},\star\boldsymbol{\rho}] &=& -[\boldsymbol{\eta},\boldsymbol{\rho}] \end{array} \tag{1.49}$$

and we denote the projectors onto the self dual and anti-selfdual part of the Lie algebra. Thanks to (1.49), we obtain:

$$\begin{array}{rcl} \mathfrak{P}^{\lhd}[\boldsymbol{\eta},\boldsymbol{\rho}] &=& [\mathfrak{P}^{\lhd}\boldsymbol{\eta},\boldsymbol{\rho}] = [\boldsymbol{\eta},\mathfrak{P}^{\lhd}\boldsymbol{\rho}] = [\mathfrak{P}^{\lhd}\boldsymbol{\eta},\mathfrak{P}^{\lhd}\boldsymbol{\rho}] \\ \mathfrak{P}^{\rhd}[\boldsymbol{\eta},\boldsymbol{\rho}] &=& [\mathfrak{P}^{\rhd}\boldsymbol{\eta},\boldsymbol{\rho}] = [\boldsymbol{\eta},\mathfrak{P}^{\rhd}\boldsymbol{\rho}] = [\mathfrak{P}^{\rhd}\boldsymbol{\eta},\mathfrak{P}^{\rhd}\boldsymbol{\rho}] \end{array} \tag{1.50}$$

This is why we can write:

$$\mathfrak{so}(1,3)_{\mathbb{C}} = \mathfrak{so}(1,3)^{\lhd}_{\mathbb{C}} \oplus \mathfrak{so}(1,3)^{\rhd}_{\mathbb{C}}$$

so that the two subspaces of $\mathfrak{so}(1,3)_{\mathbb{C}}$ defined respectively by $\mathfrak{P}^{\lhd}$ and $\mathfrak{P}^{\rhd}$ are ideals of the Lie algebra. We define, the self-dual connection $\omega^{\lhd}$ and the anti self dual connection $\omega^{\rhd}$ as:

$$\left|\begin{array}{rcl} \omega^{\lhd} &=& \mathfrak{P}^{\lhd}\omega = (1/2)\big[\omega - i\star\omega\big] \\ \omega^{\rhd} &=& \mathfrak{P}^{\rhd}\omega = (1/2)\big[\omega + i\star\omega\big] \end{array}\right. \tag{1.51}$$

$$\begin{array}{rcl} \star[\boldsymbol{\eta},\star\boldsymbol{\rho}]^{IJ} &=& \frac{1}{2}\varepsilon^{IJ}{}_{MN}\big[\boldsymbol{\eta}^M{}_K\big[\frac{1}{2}\varepsilon^{KN}{}_{QR}\boldsymbol{\rho}^{QR}\big] - \big[\frac{1}{2}\eta_{SK}\varepsilon^{MS}{}_{QR}\boldsymbol{\rho}^{QR}\big]\boldsymbol{\eta}^{KN}\big] \\ &=& (1/4)\varepsilon^{IJ}{}_{MN}\big[\varepsilon^{KN}{}_{QR}\boldsymbol{\eta}^M{}_K\boldsymbol{\rho}^{QR} - \eta_{SK}\varepsilon^{MS}{}_{QR}\boldsymbol{\rho}^{QR}\boldsymbol{\eta}^{KN}\big] \\ &=& (1/4)\varepsilon^{IJ}{}_{MN}\big[\varepsilon^{KN}{}_{QR}\boldsymbol{\eta}^M{}_K\boldsymbol{\rho}^{QR} - \varepsilon^M{}_{KQR}\boldsymbol{\rho}^{QR}\boldsymbol{\eta}^{KN}\big] \\ &=& (1/4)\big[\varepsilon^{IJ}{}_{MN}\varepsilon^{KN}{}_{QR}\boldsymbol{\eta}^M{}_K - \varepsilon^{IJ}{}_{MN}\varepsilon^M{}_{KQR}\boldsymbol{\eta}_T{}^N\eta^{TK}\big]\boldsymbol{\rho}^{QR} \\ &=& (1/4)\big[\varepsilon^{IJ}{}_{MN}\varepsilon^{KN}{}_{QR}\boldsymbol{\eta}^M{}_K - \varepsilon^{IJ}{}_{MN}\varepsilon^{MT}{}_{QR}\boldsymbol{\eta}_T{}^N\big]\boldsymbol{\rho}^{QR} \\ &=& (1/4)\big[\varepsilon^{IJ}{}_{MN}\varepsilon^{KN}{}_{QR}\boldsymbol{\eta}^M{}_K - \varepsilon^{IJ}{}_{MN}\varepsilon^{MK}{}_{QR}\boldsymbol{\eta}_K{}^N\big]\boldsymbol{\rho}^{QR} \\ &=& (1/4)\big[\varepsilon^{IJ}{}_{MN}\varepsilon^{KN}{}_{QR}\boldsymbol{\eta}^M{}_K + \varepsilon^{IJ}{}_{MN}\varepsilon^{MK}{}_{QR}\boldsymbol{\eta}^N{}_K\big]\boldsymbol{\rho}^{QR} \\ &=& (1/4)\big[\varepsilon^{IJ}{}_{MN}\varepsilon^{KN}{}_{QR}\boldsymbol{\eta}^M{}_K + \varepsilon^{IJ}{}_{NM}\varepsilon^{NK}{}_{QR}\boldsymbol{\eta}^M{}_K\big]\boldsymbol{\rho}^{QR} \\ &=& (1/4)\big[\varepsilon^{IJ}{}_{MN}\varepsilon^{KN}{}_{QR} + \varepsilon^{IJ}{}_{NM}\varepsilon^{NK}{}_{QR}\big]\boldsymbol{\eta}^M{}_K\boldsymbol{\rho}^{QR} \\ &=& (1/2)\big[\varepsilon^{IJ}{}_{MN}\varepsilon^{KN}{}_{QR}\big]\boldsymbol{\eta}^M{}_K\boldsymbol{\rho}^{QR} \end{array}$$

Finally, we end the calculation via:

$$\begin{array}{rcl} \star[\boldsymbol{\eta},\star\boldsymbol{\rho}]^{IJ} &=& (1/2)\big[(\varepsilon^{IJCD}\eta_{CM}\eta_{DN})(\eta^{AK}\eta^{BN}\varepsilon_{ABQR})\big]\boldsymbol{\eta}^M{}_K\boldsymbol{\rho}^{QR} \\ &=& (1/2)\big[(\varepsilon^{IJCD}\varepsilon_{ABQR})(\eta_{CM}\eta_{DN})(\eta^{AK}\eta^{BN})\big]\boldsymbol{\eta}^M{}_K\boldsymbol{\rho}^{QR} \\ &=& (1/2)\big[(\varepsilon^{IJCD}\varepsilon_{ABQR})(\eta_{CM}\eta^{AK})(\eta_{DN}\eta^{BN})\big]\boldsymbol{\eta}^M{}_K\boldsymbol{\rho}^{QR} \\ &=& (1/2)\big[(\varepsilon^{IJCD}\varepsilon_{ABQR})(\eta_{DN}\eta^{BN})\big]\boldsymbol{\eta}_C{}^A\boldsymbol{\rho}^{QR} \\ &=& (1/2)\big[(\varepsilon^{IJMD}\varepsilon_{KBQR})(\delta^B_D)\big]\boldsymbol{\eta}_M{}^K\boldsymbol{\rho}^{QR} \\ &=& (1/2)\big[(\varepsilon^{IJMD}\varepsilon_{KQRD})\big]\boldsymbol{\eta}_M{}^K\boldsymbol{\rho}^{QR} \\ &=& -(1/2)\big[(\varepsilon^{IJMD}\varepsilon_{KQRD})\big]\boldsymbol{\eta}^K{}_M\boldsymbol{\rho}^{QR} \end{array}$$

Notice that since:

$$-(1/2)\big[(\varepsilon^{IJMD}\varepsilon_{KQRD})\big]\boldsymbol{\eta}^K{}_M\boldsymbol{\rho}^{QR} = -(1/2)\big[(-6\delta^I_{[K}\delta^J_Q\delta^M_{R]})\big]\boldsymbol{\eta}^K{}_M\boldsymbol{\rho}^{QR}$$

we have:

$$\star[\boldsymbol{\eta},\star\boldsymbol{\rho}]^{IJ} = (1/2)\big[\delta^I_K\delta^J_Q\delta^M_R + \delta^I_R\delta^J_K\delta^M_Q + \delta^I_Q\delta^J_R\delta^M_K - \delta^I_Q\delta^J_K\delta^M_R - \delta^I_R\delta^J_Q\delta^M_K - \delta^I_K\delta^J_R\delta^M_Q\big]\boldsymbol{\eta}^K{}_M\boldsymbol{\rho}^{QR}$$

so that:

$$\begin{array}{rcl} \star[\boldsymbol{\eta},\star\boldsymbol{\rho}]^{IJ} &=& (1/2)\Big[\boldsymbol{\eta}^I{}_M\boldsymbol{\rho}^{JM} + \boldsymbol{\eta}^J{}_M\boldsymbol{\rho}^{MI} + \boldsymbol{\eta}^M{}_M\boldsymbol{\rho}^{IJ} - \boldsymbol{\eta}^J{}_M\boldsymbol{\rho}^{IM} - \boldsymbol{\eta}^M{}_M\boldsymbol{\rho}^{JI} - \boldsymbol{\eta}^I{}_M\boldsymbol{\rho}^{MJ}\Big] \\ &=& -(1/2)\Big[-\boldsymbol{\eta}^I{}_K\boldsymbol{\rho}^{JK} - \boldsymbol{\eta}^J{}_K\boldsymbol{\rho}^{KI} + \boldsymbol{\eta}^J{}_K\boldsymbol{\rho}^{IK} + \boldsymbol{\eta}^I{}_K\boldsymbol{\rho}^{KJ}\Big] \end{array} \tag{1.48}$$

which is equivalently written; $\star[\boldsymbol{\eta},\star\boldsymbol{\rho}] = \star[\boldsymbol{\eta},\star\boldsymbol{\rho}]^{IJ}\otimes\boldsymbol{\Delta}_{IJ}$, so that: $\star[\boldsymbol{\eta},\star\boldsymbol{\rho}]^{IJ} = \Big[\boldsymbol{\eta}^I{}_K\boldsymbol{\rho}^{KJ} - \boldsymbol{\rho}^I{}_K\boldsymbol{\eta}^{KJ}\Big] = -[\boldsymbol{\eta},\boldsymbol{\rho}]^{IJ}$

So that we decompose $\omega = \omega^{\lhd} + \omega^{\rhd} = \mathfrak{P}^{\lhd}\omega + \mathfrak{P}^{\rhd}\omega$. Notice that, the self dual part of the curvature 2-form is written:

$$\left|\begin{array}{lcl} (\mathsf{F}^{\lhd})^{IJ}_{\mu\nu} & = & (\omega^{\lhd})^{IJ}_{[\mu,\nu]} + (\omega^{\lhd})^{I}{}_{[\mu K}(\omega^{\lhd})^{KJ}{}_{\nu]} \\ (\mathsf{F}^{\lhd})_{\mu\nu} & = & \mathfrak{P}^{\lhd}\omega_{[\mu,\nu]} + \frac{1}{2}[\mathfrak{P}^{\lhd}\omega_{\mu}, \mathfrak{P}^{\lhd}\omega_{\nu}] = \mathfrak{P}^{\lhd}\omega_{[\mu,\nu]} + \mathfrak{P}^{\lhd}\frac{1}{2}[\omega_{\mu},\omega_{\nu}] = \mathfrak{P}^{\lhd}\mathsf{F}_{\mu\nu} \end{array}\right. \tag{1.52}$$

and the anti self dual part of the curvature 2-form is written

$$\left|\begin{array}{lcl} (\mathsf{F}^{\rhd})^{IJ}_{\mu\nu} & = & (\omega^{\rhd})^{IJ}_{[\mu,\nu]} + (\omega^{\rhd})^{I}{}_{[\mu K}(\omega^{\rhd})^{KJ}{}_{\nu]} \\ (\mathsf{F}^{\rhd})_{\mu\nu} & = & \mathfrak{P}^{\rhd}\omega_{[\mu,\nu]} + \frac{1}{2}[\mathfrak{P}^{\rhd}\omega_{\mu}, \mathfrak{P}^{\rhd}\omega_{\nu}] = \mathfrak{P}^{\rhd}\omega_{[\mu,\nu]} + \mathfrak{P}^{\rhd}\frac{1}{2}[\omega_{\mu},\omega_{\nu}] = \mathfrak{P}^{\rhd}\mathsf{F}_{\mu\nu} \end{array}\right. \tag{1.53}$$

Notice also that

$$\mathfrak{P}^{\lhd}[\boldsymbol{\eta},\boldsymbol{\rho}]^{IJ} = \frac{1}{2}\left[[\boldsymbol{\eta},\boldsymbol{\rho}]^{IJ} - i\star[\boldsymbol{\eta},\boldsymbol{\rho}]^{IJ}\right] = \frac{1}{2}\left[[\boldsymbol{\eta},\boldsymbol{\rho}]^{IJ} + [(-i\star)\boldsymbol{\eta},\boldsymbol{\rho}]^{IJ}\right]$$

sot that: $\mathfrak{P}^{\lhd}[\boldsymbol{\eta},\boldsymbol{\rho}]^{IJ} = [\frac{1}{2}\left[\boldsymbol{\eta} - i\star\boldsymbol{\eta}\right],\boldsymbol{\rho}]^{IJ} = [\mathfrak{P}^{\lhd}\boldsymbol{\eta},\boldsymbol{\rho}]^{IJ}$. We summarize the properties

$$\left|\begin{array}{rcl} \mathfrak{P}^{\lhd}[\boldsymbol{\eta},\boldsymbol{\rho}]^{IJ} & = & [\mathfrak{P}^{\lhd}\boldsymbol{\eta},\boldsymbol{\rho}]^{IJ} = [\boldsymbol{\eta},\mathfrak{P}^{\lhd}\boldsymbol{\rho}]^{IJ} \\ \mathfrak{P}^{\rhd}[\boldsymbol{\eta},\boldsymbol{\rho}]^{IJ} & = & [\mathfrak{P}^{\rhd}\boldsymbol{\eta},\boldsymbol{\rho}]^{IJ} = [\boldsymbol{\eta},\mathfrak{P}^{\rhd}\boldsymbol{\rho}]^{IJ} \\ [\mathfrak{P}^{\lhd}\boldsymbol{\eta},\mathfrak{P}^{\rhd}\boldsymbol{\rho}] & = & [\mathfrak{P}^{\rhd}\boldsymbol{\eta},\mathfrak{P}^{\lhd}\boldsymbol{\rho}] = 0 \end{array}\right. \tag{1.54}$$

1.3.4 Self-dual action

Notice that: we have the following isomorphism or decomposition:

$$\mathfrak{so}(1,3)_{\mathbb{C}} = \mathfrak{so}(1,3)\otimes\mathbb{C} = \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}) \tag{1.55}$$

Notice that beyond this decomposition is find, the relation with the original work of Plebanski and Ashtekar. For details on this decomposition and the $\mathfrak{sl}(2,\mathbb{C})$-spin variables, we refer to the section (1.4) - for group and Lie algebra isomorphism and complexification of Lie algebra - or to section (2) for the self-dual-solutions of Einstein's Equation via the work of Plebanski and Ashtekar.

$$\mathfrak{so}(1,3)_{\mathbb{C}} = \mathfrak{so}(1,3)^{\lhd}_{\mathbb{C}}\oplus\mathfrak{so}(1,3)^{\rhd}_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$$

with:

$$\begin{array}{lcl} \mathfrak{so}(1,3)^{\lhd}_{\mathbb{C}} & = & \left\{\boldsymbol{\eta}\in\Omega^2(\mathcal{X})\otimes\mathfrak{so}(1,3)\ /\ \star\boldsymbol{\eta} = i\boldsymbol{\eta}\ \text{Self Dual}\right\} \\ \mathfrak{so}(1,3)^{\rhd}_{\mathbb{C}} & = & \left\{\boldsymbol{\eta}\in\Omega^2(\mathcal{X})\otimes\mathfrak{so}(1,3)\ /\ \star\boldsymbol{\eta} = -i\boldsymbol{\eta}\ \text{Anti Self Dual}\right\} \end{array} \tag{1.56}$$

We decompose the action into self dual and anti-self-dual parts: $\mathcal{L}_{\mathsf{EC}}[e,\omega] = \mathcal{L}^{\lhd}_{\mathsf{EC}}[e,\omega^{\lhd}] + \mathcal{L}^{\rhd}_{\mathsf{EC}}[e,\omega^{\rhd}] = \mathcal{L}^{\lhd}_{\mathsf{EC}} + \mathcal{L}^{\rhd}_{\mathsf{EC}}$, where the self dual part $\mathcal{L}^{\lhd}_{\mathsf{EC}}$ and the anti-self-dual part $\mathcal{L}^{\rhd}_{\mathsf{EC}}$ are respectively described as:

$$\left|\begin{array}{lcl} \mathcal{L}^{\lhd}_{\mathsf{EC}}[e,\omega^{\lhd}] & = & \mathsf{F}^{\lhd}_{IJ}\wedge\star(e^I\wedge e^J) \\ \mathcal{L}^{\rhd}_{\mathsf{EC}}[e,\omega^{\rhd}] & = & \mathsf{F}^{\rhd}_{IJ}\wedge\star(e^I\wedge e^J) \end{array}\right. \tag{1.57}$$

so that

$$\begin{array}{rcl} \mathcal{L}_{\mathsf{EC}}[e,\omega] & = & \int\left[\mathsf{F}^{\lhd}_{IJ} + \mathsf{F}^{\rhd}_{IJ}\right]\wedge\star(e^I\wedge e^J) = \int\mathsf{F}^{\lhd}_{IJ}\wedge\star(e^I\wedge e^J) + \int\mathsf{F}^{\rhd}_{IJ}\wedge\star(e^I\wedge e^J) \\ & = & \int\star\left[\mathsf{F}^{\lhd}_{IJ}\right]\wedge(e^I\wedge e^J) + \int\star\left[\mathsf{F}^{\rhd}_{IJ}\right]\wedge(e^I\wedge e^J) \\ & = & i\int\mathsf{F}^{\lhd}_{IJ}\wedge(e^I\wedge e^J) - i\int\mathsf{F}^{\rhd}_{IJ}\wedge(e^I\wedge e^J) \end{array} \tag{1.58}$$

with

$$\mathsf{F}^{KL} = \underbrace{\left[\mathsf{F}^{\triangleleft}\right]^{KL}}_{\text{Self-Dual}} + \underbrace{\left[\mathsf{F}^{\triangleright}\right]^{KL}}_{\text{Anti-Self-Dual}} = \underbrace{\frac{1}{2}\left[\mathsf{F}^{IJ} - i\star(\mathsf{F}^{IJ})\right]}_{\text{Self-Dual}} + \underbrace{\frac{1}{2}\left[\mathsf{F}^{IJ} + i\star(\mathsf{F}^{IJ})\right]}_{\text{Anti-Self-Dual}}$$

For example the anti-selfdual part is written: $\underbrace{\left[\mathsf{F}^{\triangleright}\right]^{KL}}_{\text{Anti-Self-Dual}} = \frac{1}{2}\left[\mathsf{F}^{IJ} + \frac{i}{2}\varepsilon^{IJ}{}_{KL}\mathsf{F}^{KL}\right]$ so that:

$$\begin{aligned} \star\left[\mathsf{F}^{\triangleright}\right] &= \star\left[\frac{1}{2}\left[\mathsf{F} + i\star(\mathsf{F})\right]\right] = \frac{-i}{2}(i)\left[\star\mathsf{F} + i\star\star\mathsf{F}\right] = \frac{-i}{2}\left[i\star\mathsf{F} + i^2\star\star\mathsf{F}\right] \\ &= \frac{-i}{2}\left[i\star\mathsf{F} - \star\star\mathsf{F}\right] = \frac{-i}{2}\left[i\star\mathsf{F} + \mathsf{F}\right] = (-i)\frac{1}{2}\left[\mathsf{F} + i\star\mathsf{F}\right] \end{aligned} \tag{1.59}$$

which leads to the result $\star\left[\mathsf{F}^{\triangleright}\right] = (-i)\mathsf{F}^{\triangleright}$, namely $\mathsf{F}^{\triangleright}$ is anti self dual. For the self dual case, where the object of interest $\left[\mathsf{F}^{\triangleleft}\right]^{KL}$ is this time self-dual: $\star\left[\mathsf{F}^{\triangleleft}\right] = i\mathsf{F}^{\triangleleft}$.

1.4 Group and Lie algebra prolegomena

Below we make the following clarification: The *real* Ashtekar connection is a connection on the internal spinor bundle over space - an $\mathsf{SU}(2)$ bundle. Now, the *complex* Ashtekar connection is a connection on the restriction to space of the left-handed internal spinor bundle over spacetime. This is an $\mathsf{SL}(2,\mathbb{C})$ bundle. In the context, of a 4D Lorentzian manifold, (namely equipped with a Lorentzian metric) we can construct a *spin structure*: a bundle with gauge group $\mathsf{SL}(2,\mathbb{C})$, the double cover of the Lorentz group. We then complexify the spin structure, the bundle obtained is the complexification of $\mathsf{SL}(2,\mathbb{C})$, denoted:

$$\mathsf{SL}(2,\mathbb{C})_{\mathbb{C}} = \mathsf{SL}(2,\mathbb{C}) \otimes \mathbb{C} = \mathsf{SL}(2,\mathbb{C}) \oplus \mathsf{SL}(2,\mathbb{C}) \tag{1.60}$$

This bundle is a complex spin bundle, the sum of two bundles with gauge group $\mathsf{SL}(2,\mathbb{C})$: the left-handed and right-handed spin bundles.

1.4.1 The group SO$(1,3)$ and SO(4)

In group theory we know that the group $\mathsf{SL}(2,\mathbb{R})$ and $\mathsf{SU}(2)$ are respectively the double covers of the groups $\mathsf{SO}(1,3)$ and $\mathsf{SO}(4)$, we can perform the following decomposition:

$$\mathsf{SO}(1,3) = \mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R}) = \underbrace{\left[\mathsf{SL}(2,\mathbb{R})\right]^{\triangleleft}}_{\text{Left-handed}} \times \underbrace{\left[\mathsf{SL}(2,\mathbb{R})\right]^{\triangleright}}_{\text{RIeght-handed}} \tag{1.61}$$

Notice that the analogue in the Euclidian case is described by:

$$\mathsf{SO}(4) = \mathsf{SU}(2) \times \mathsf{SU}(2) = \left[\mathsf{SU}(2)\right]^{\triangleleft} \times \left[\mathsf{SU}(2)\right]^{\triangleright} \tag{1.62}$$

1.4.2 The group SU(2) and its Lie algebra $\mathfrak{su}(2)$

The group $\mathsf{SU}(2)$ is defined to be:

$$\mathsf{SU}(2) = -\left\{\mathsf{M} \in \mathcal{M}_2(\mathbb{C}) \;/\; \det(\mathsf{M}) = 1 \text{ and } \mathsf{M}\mathsf{M}^{\dagger} = 1\right\} \tag{1.63}$$

whereas the Lie algebra $\mathfrak{su}(2)$

$$\mathfrak{su}(2) = -\left\{\mathsf{M} \in \mathcal{M}_2(\mathbb{C}) \;/\; \mathrm{Tr}(\mathsf{M}) = 0 \text{ and } \mathsf{M}^{\dagger} = -\mathsf{M}\right\} \tag{1.64}$$

The Lie algebra $\mathfrak{su}(2)$ is a real Lie algebra. We observe that $\dim(\mathfrak{su}(2)) = 3$. Notice that in that case, we consider a different way of writting this, with the basis element are written: $\mathsf{T}_j = -\frac{i}{2}\boldsymbol{\sigma}_j$, where the $\boldsymbol{\sigma}_j$ are the Pauli matrices:

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.65}$$

so that

$$\mathsf{T}_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \qquad \mathsf{T}_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{T}_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{1.66}$$

which is known as the fundamental representation. We have the commutation relation described by: $[\mathsf{T}_i, \mathsf{T}_j] = \varepsilon_{ij}{}^k \mathsf{T}_k$, so that (Hence, real linear combinations of T_j lead us to consider the real Lie algebra $\mathfrak{su}(2)$.):

$$\begin{aligned} [\mathsf{T}_1, \mathsf{T}_2] &= \mathsf{T}_3 \\ [\mathsf{T}_2, \mathsf{T}_3] &= \mathsf{T}_1 \\ [\mathsf{T}_3, \mathsf{T}_1] &= \mathsf{T}_2 \end{aligned} \tag{1.67}$$

1.4.3 Complexified Lie algebra $\mathfrak{so}(1,3) \otimes \mathbb{C}$

Recall that if we denote $\boldsymbol{\Delta}_i$ a basis of the Lie algebre $\mathfrak{g}$, then the complexified Lie algebra $\mathfrak{g} \otimes \mathbb{C}$ is considered as $\left\{\alpha^i \boldsymbol{\Delta}_i \ / \ \alpha^i \in \mathbb{C}\right\}$. Notice that a representation of a real Lie algebra can be given with matrices with complex coefficients. The Lie algebra is real if its structure constant are real. In turn, the complexification of the real Lie algebra $\mathfrak{g}$ is denoted $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$. The orthonormal frame bundle has structure group $\mathsf{SO}(1,3)$ which have a simple Lie algebra, its complexification $\mathfrak{so}(1,3)_{\mathbb{C}} = so(1,3) \otimes \mathbb{C}$ is the complexified $\mathfrak{so}(1,3)_{\mathbb{C}}$ lie algebra and we decompose it:

$$\mathfrak{so}(1,3)_{\mathbb{C}} = so(1,3) \otimes \mathbb{C} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \tag{1.68}$$

We construct each copies $\mathfrak{sl}(2,\mathbb{C})$ of the complexified Lie algebra $\mathfrak{so}(1,3)_{\mathbb{C}} = so(1,3) \otimes \mathbb{C}$ - perceived as the complexified Lie algebra of $\mathfrak{s}o(3)$, namely $\mathfrak{so}(3) \otimes \mathbb{C}$ - or equivalently $\mathfrak{sl}(2,\mathbb{C})$ is perceived as the complexified Lie algebra of $\mathfrak{s}u(2)$ that is $\mathfrak{su}(2) \otimes \mathbb{C}$:

$$\mathfrak{sl}(2,\mathbb{C}) \ \cong \ \mathfrak{so}(3) \otimes \mathbb{C} \cong \mathfrak{su}(2) \otimes \mathbb{C} \tag{1.69}$$

We also notice that $\mathsf{Spin}(1,3) \cong \mathsf{SL}(2,\mathbb{C})$. The group $\mathsf{Spin}(1,3)$ is the double cover of the group $\mathsf{SL}(2,\mathbb{C})$ This isomorphism, allows to make connection with the spinor representation, via the isomorphism of group: $\mathsf{Spin}(1,3) \otimes \mathbb{C} \cong \mathsf{SL}(2,\mathbb{C}) \times \mathsf{SL}(2,\mathbb{C})$ and introduce the 2-dimensional complex vector bundle: the self dual $\mathfrak{S}^{\lhd}$ and antiself dual spinors bundle $\mathfrak{S}^{\rhd}$, which are perceived as $\mathsf{SL}(2,\mathbb{C})$ bundles.

1.4.4 The group $\mathsf{SL}(2,\mathbb{C})$

Notice that the group $\mathsf{SL}(2,\mathbb{C})$ is the group of non-degenerate complexe matrices with unit determinant. It is a 3-dimensional complex Lie group - $\dim_{\mathbb{C}}(\mathsf{SL}(2,\mathbb{C})) = 3$ - or a 6-dimensional real Lie group, $\dim_{\mathbb{R}}(\mathsf{SL}(2,\mathbb{C})) = 6$.

$$\mathsf{SL}(2,\mathbb{C}) = \Big\{\mathsf{M} \in \mathcal{M}_2(\mathbb{C}) \ / \ \det(\mathsf{M}) = 1\Big\} = \Big\{\mathsf{M} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \ / \ \alpha, \beta, \gamma, \delta \in \mathbb{C}, \ \alpha\delta - \beta\gamma = 1\Big\}$$

and: $\dim(\mathsf{SL}(2,\mathbb{R})) = \dim(\mathsf{SU}(2) = 3$. Note also that the Lie algebra

$$\mathfrak{sl}(2,\mathbb{C}) = \Big\{\mathsf{M} \in \mathcal{M}_2(\mathbb{C}) \;/\; \mathrm{tr}(\mathsf{M}) = 0\Big\}$$

Since $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \otimes \mathbb{C}$, the same matrices can also be taken as a basis of $\mathfrak{sl}(2,\mathbb{C})$. A basis for $\mathfrak{sl}(2,\mathbb{C})$ as a $\mathbb{C}$-vector space is denoted:

$$\mathsf{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \mathsf{F} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The basis $\mathsf{E}, \mathsf{F}, \mathsf{H}$ is the Cartan-Weyl basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. The Lie bracket given by

$$\begin{aligned} [\mathsf{H},\mathsf{E}] &= 2\mathsf{E} \\ [\mathsf{H},\mathsf{F}] &= -2\mathsf{F} \\ [\mathsf{E},\mathsf{F}] &= \mathsf{H} \end{aligned} \tag{1.70}$$

If we consider real linear combinations of E, F and H, we obtain the real Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Now if we consider complex linear combinations of E, F and H, we find the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. We picture different representation for the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. For example, considering it as a real vectorial space, we choose the following basis $\big\{\mathsf{T}_j, \mathsf{P}_j\big\}_{1\leq j\leq 3}$ with, $\mathsf{T}_j = -\frac{i}{2}\boldsymbol{\sigma}_j$ and $\mathsf{P}_j = -\frac{1}{2}\boldsymbol{\sigma}_j$

$$\mathsf{T}_1 = -\frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \mathsf{T}_2 = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathsf{T}_3 = -\frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\mathsf{P}_1 = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{P}_2 = -\frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathsf{P}_3 = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Notice that T_j (hence, $\mathsf{T}_j^\dagger = -\mathsf{T}_j$) are anti-Hermitian and P_j are Hermitian (namely $\mathsf{P}_j^\dagger = \mathsf{P}_j$). The Lie algebra is denoted:

$$\begin{aligned} [\mathsf{T}_i,\mathsf{T}_j] &= \varepsilon_{ij}{}^k\mathsf{P}_k \\ [\mathsf{P}_i,\mathsf{T}_j] &= \varepsilon_{ij}{}^k\mathsf{P}_k \\ [\mathsf{P}_i,\mathsf{P}_j] &= -\varepsilon_{ij}{}^k\mathsf{T}_k \end{aligned}$$

We have:

$$\begin{aligned} [\mathsf{T}_1,\mathsf{T}_2] &= \mathsf{P}_3 & [\mathsf{P}_1,\mathsf{T}_2] &= \mathsf{P}_3 & [\mathsf{T}_1,\mathsf{P}_2] &= \mathsf{P}_3 & [\mathsf{P}_1,\mathsf{P}_2] &= -\mathsf{T}_3 \\ [\mathsf{T}_2,\mathsf{T}_3] &= \mathsf{P}_1 & [\mathsf{P}_2,\mathsf{T}_3] &= \mathsf{P}_1 & [\mathsf{T}_2,\mathsf{P}_3] &= \mathsf{P}_1 & [\mathsf{P}_2,\mathsf{P}_3] &= -\mathsf{T}_1 \\ [\mathsf{T}_3,\mathsf{T}_1] &= \mathsf{P}_2 & [\mathsf{P}_3,\mathsf{T}_1] &= \mathsf{P}_2 & [\mathsf{T}_3,\mathsf{P}_1] &= \mathsf{P}_2 & [\mathsf{P}_3,\mathsf{P}_1] &= -\mathsf{T}_2 \end{aligned}$$

Subalgebra $\mathfrak{su}(2)$ is defined by the matrices T_i.[19] Notice that we have an isomorphism $\mathfrak{su}(2)_\mathbb{C} = \mathfrak{su}(2) \otimes \mathbb{C} \cong \mathfrak{sl}(2,\mathbb{C})$ via the definition of a new basis T^+ and T^-

$$\mathsf{T}_j^+ = \frac{1}{2}(\mathsf{T}_j + i\mathsf{P}_j) \qquad \mathsf{T}_j^- = \frac{1}{2}(\mathsf{T}_j - i\mathsf{P}_j)$$

so that we find the commutations relations in the following form:

$$\begin{aligned} [\mathsf{T}_i^+,\mathsf{T}_j^+] &= \varepsilon_{ij}{}^k\mathsf{T}_k^+ \\ [\mathsf{T}_i^-,\mathsf{T}_j^-] &= \varepsilon_{ij}{}^k\mathsf{T}_k^- \\ [\mathsf{T}_i^+,\mathsf{T}_j^-] &= 0 \end{aligned}$$

In such a case, this decomposition is equivalent to the isomorphism of Lie algebra: $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) + i\mathfrak{su}(2)$.

[19] We can also consider the subalgebra defined by the real traceless matrices $\mathbf{C}_1 = \mathsf{T}_1, \mathbf{C}_2 = \mathsf{P}_2$ and $\mathbf{C}_3 = \mathsf{T}_3$ so that we have the following commutation relations: $[\mathbf{C}_1,\mathbf{C}_2]\mathbf{C}_3$, $[\mathbf{C}_2,\mathbf{C}_3] = \mathbf{C}_1$ and $[\mathbf{C}_3,\mathbf{C}_1] = -\mathbf{C}_2$ and is related to the Lie algebra of $\mathsf{SL}(2,\mathbb{R})$ following the decomposition. $\mathsf{SO}(1,3) = \mathsf{SL}(2,\mathbb{R})\times\mathsf{SL}(2,\mathbb{R}) = \big[\mathsf{SL}(2,\mathbb{R})\big]^{\triangleleft}\times\big[\mathsf{SL}(2,\mathbb{R})\big]^{\triangleright}$.

1.5 Geometry of bivectors and 2-forms

1.5.1 $\mathfrak{U}_\otimes = e\wedge e \in \Omega^2(\mathcal{X})\otimes \boldsymbol{\mathcal{V}}^q_\otimes$ and $\mathfrak{U}_\wedge = e \boldsymbol{\wedge} e \in \Omega^2(\mathcal{X})\otimes \boldsymbol{\mathcal{V}}^q_\wedge$

Finally, the last case of interest is when $\boldsymbol{\varphi} \in \Omega^p(\mathcal{X},\mathcal{V})$ and $\boldsymbol{\eta} \in \Omega^q(\mathcal{X},\mathcal{V})$.[20]

$$\boldsymbol{\varphi} \wedge \boldsymbol{\eta}(\zeta_1 ... \zeta_{p+q}) \;=\; \frac{1}{p!q!}\sum_{\sigma\in\mathcal{S}_{p+q}} (-1)^{\boldsymbol{\sigma}}(\boldsymbol{\varphi}(\zeta_{\sigma(1)}...\zeta_{\sigma(p)}))\otimes(\boldsymbol{\eta}(\zeta_{\sigma(p+1)}...\zeta_{\sigma(p+q)})) \tag{1.71}$$

The wedge product is understood as an operation: $\wedge : \Omega^p(\mathcal{X},\mathcal{V})\times\Omega^q(\mathcal{X},\mathcal{V}) \to \Omega^{p+q}(\mathcal{X},\mathcal{V}\otimes\mathcal{V})$. Notice that $\boldsymbol{\varphi}\wedge\boldsymbol{\eta}(\zeta_1...\zeta_{p+q}) \in \mathcal{V}\otimes\mathcal{V}$ and that we can write for a basis $\mathfrak{e}_I$ of $\mathcal{V}$, the following forms $\boldsymbol{\varphi} = \boldsymbol{\varphi}^I\mathfrak{e}_I$ and $\boldsymbol{\eta}^J\mathfrak{e}_J$, so that: $\boldsymbol{\varphi}\wedge\boldsymbol{\eta} = \sum_{I,J}\boldsymbol{\varphi}^I\wedge\boldsymbol{\eta}^J\mathfrak{e}_I\otimes\mathfrak{e}_J$. Notice, that in this notation, we write:

$$\begin{aligned}\boldsymbol{\varphi} &= \boldsymbol{\varphi}^I\mathfrak{e}_I = (1/p!)\boldsymbol{\varphi}^I_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\mathrm{d}x^{\mu_p}\otimes\mathfrak{e}_I\\ \boldsymbol{\eta} &= \boldsymbol{\eta}^J\mathfrak{e}_J = (1/q!)\boldsymbol{\eta}^I_{\mu_1\cdots\mu_q}\mathrm{d}x^{\mu_1}\wedge\cdots\mathrm{d}x^{\mu_q}\otimes\mathfrak{e}_J\end{aligned} \tag{1.72}$$

so that for any $\zeta_1,\cdots,\zeta_{p+q}\in T\mathcal{X}$

$$\begin{aligned}\boldsymbol{\varphi}\wedge\boldsymbol{\eta}(\zeta_1,\cdots,\zeta_{p+q}) &= \big[\frac{1}{p!}\boldsymbol{\varphi}^I_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\mathrm{d}x^{\mu_p}\otimes\mathfrak{e}_I\big]\wedge\big[\frac{1}{q!}\boldsymbol{\eta}^I_{\mu_1\cdots\mu_q}\mathrm{d}x^{\mu_1}\wedge\cdots\mathrm{d}x^{\mu_q}\otimes\mathfrak{e}_J\big](\zeta_1,\cdots,\zeta_{p+q})\\ &= \frac{1}{p!q!}\sum_{\sigma\in\mathcal{S}_{p+q}}(-1)^{\boldsymbol{\sigma}}(\boldsymbol{\varphi}(\zeta_{\sigma(1)}...\zeta_{\sigma(p)}))\otimes(\boldsymbol{\eta}(\zeta_{\sigma(p+1)}...\zeta_{\sigma(p+q)}))\end{aligned}$$

We consider now the Minkowski case where $\mathcal{V}=\mathbb{R}^{1,3}$. Let $\mathfrak{e}_I$ be a basis of the Minkowski vector space $\mathbb{R}^{1,3}$ and we denote $\mathfrak{e}_I\otimes\mathfrak{e}_J$ a basis of the space $\mathcal{V}\otimes\mathcal{V}$. We denote $\boldsymbol{\mathcal{V}}^q_\otimes = \bigotimes_{1\le j\le q}\mathcal{V}$.

1.5.2 The form $\mathfrak{U}_\otimes = e\wedge e \in \Omega^2(\mathcal{X})\otimes\boldsymbol{\mathcal{V}}^q_\otimes$

If we apply the wedge product (1.71) for two forms $e = \mathfrak{e}_I e^I_\mu \mathrm{d}x^\mu = e^I_\mu\mathrm{d}x^\mu\otimes\mathfrak{e}_I \in \Omega^1(\mathcal{X})\otimes\mathcal{V}\cong\Omega^1(\mathcal{X})\otimes\mathbb{R}^{1,3}$ so that the object $e\wedge e$ is written: $e\wedge e = (\mathfrak{e}_I e^I)\wedge(\mathfrak{e}_J e^J)\in\Omega^2(\mathcal{X})\otimes(\mathcal{V}\otimes\mathcal{V}) = \Omega^2(\mathcal{X})\otimes\boldsymbol{\mathcal{V}}_\otimes$, we write, for any $\zeta_1,\zeta_2\in T\mathcal{X}$

$$\begin{aligned}e\wedge e(\zeta_1,\zeta_2) &= \big[e^I_\mu\mathrm{d}x^\mu\otimes\mathfrak{e}_I\big]\wedge\big[e^I_\mu\mathrm{d}x^\mu\otimes\mathfrak{e}_I\big](\zeta_1,\zeta_2)\\ &= e(\zeta_1)\otimes e(\zeta_2) - e(\zeta_2)\otimes e(\zeta_1)\\ &= \big[e^I_\mu\mathrm{d}x^\mu\otimes\mathfrak{e}_I\big](\zeta_1)\otimes\big[e^J_\nu\mathrm{d}x^\nu\otimes\mathfrak{e}_J\big](\zeta_2)\\ &\quad -\big[e^I_\mu\mathrm{d}x^\mu\otimes\mathfrak{e}_I\big](\zeta_2)\otimes\big[e^J_\nu\mathrm{d}x^\nu\otimes\mathfrak{e}_J\big](\zeta_1)\end{aligned} \tag{1.73}$$

Notice that if $\zeta_1 = \partial_\alpha$ and $\zeta_2 = \partial_\beta$

$$\begin{aligned}e\wedge e(\partial_\alpha,\partial_\beta) &= \big[e^I_\mu\mathrm{d}x^\mu\otimes\mathfrak{e}_I\big](\partial_\alpha)\otimes\big[e^J_\nu\mathrm{d}x^\nu\otimes\mathfrak{e}_J\big](\partial_\beta)\\ &\quad -\big[e^I_\mu\mathrm{d}x^\mu\otimes\mathfrak{e}_I\big](\partial_\beta)\otimes\big[e^J_\nu\mathrm{d}x^\nu\otimes\mathfrak{e}_J\big](\partial_\alpha)\\ &= e^I_\alpha e^J_\beta\mathfrak{e}_I\otimes\mathfrak{e}_J - e^I_\beta e^J_\alpha\mathfrak{e}_I\otimes\mathfrak{e}_J\\ &= \big[e^I_\alpha e^J_\beta - e^I_\beta e^J_\alpha\big]\mathfrak{e}_I\otimes\mathfrak{e}_J = 2e^{[I}_\alpha e^{J]}_\beta\mathfrak{e}_I\otimes\mathfrak{e}_J\end{aligned} \tag{1.74}$$

Hence, we describe in components:

$$e\wedge e = (e\wedge e)^{IJ}\otimes(\mathfrak{e}_I\otimes\mathfrak{e}_J) = (e\wedge e)^{IJ}\mathfrak{e}_I\otimes\mathfrak{e}_J \tag{1.75}$$

with $(e\wedge e)^{IJ} = 2e^{[I}_\alpha e^{J]}_\beta \in \mathcal{V}\otimes\mathcal{V} = \boldsymbol{\mathcal{V}}_\otimes$ which in turn is also written as:

$$\begin{aligned}e\wedge e &= \frac{1}{2}(e\wedge e)^{IJ}_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\otimes(\mathfrak{e}_I\otimes\mathfrak{e}_J)\\ &= e^{[I}_\alpha e^{J]}_\beta\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\otimes\mathfrak{e}_I\otimes\mathfrak{e}_J\end{aligned} \tag{1.76}$$

[20] We define the wedge product $\boldsymbol{\wedge}$ (bold symbol as opposed to the normal wedge product $\wedge$)

The expression (1.76) is written, with the notation $\mathfrak{U}_\otimes = e \wedge e$

$$\mathfrak{U}_\otimes = \frac{1}{2}(\mathfrak{U}_\otimes)^{IJ}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes (\mathfrak{e}_I \otimes \mathfrak{e}_J) \quad \text{with} \quad (\mathfrak{U}_\otimes)^{IJ}_{\mu\nu} = e^{[I}_\alpha e^{J]}_\beta \tag{1.77}$$

and we denote $\mathfrak{U}^{IJ}_\otimes = (e\wedge e)^{IJ} = e^I \wedge e^J$ and $\mathfrak{U}_\otimes = \mathfrak{U}^{IJ}_\otimes \otimes \mathfrak{e}_I \wedge \mathfrak{e}_J$.

1.5.3 The form $\mathfrak{U}_\wedge = e \wedge e \in \Omega^2(\mathcal{X}) \otimes \boldsymbol{\mathcal{V}}^q_\wedge$

We denote the $\boldsymbol{\mathcal{V}}^q_\wedge$-valued 2 form $\mathfrak{U}_\wedge$ $\mathfrak{U}_\wedge = e \wedge e \in \Omega^2(\mathcal{X}) \otimes \Lambda^2\mathcal{V} = \Omega^2(\mathcal{X}) \otimes \boldsymbol{\mathcal{V}}^2_\wedge$ with $\boldsymbol{\mathcal{V}}_\wedge = \Lambda^2\mathcal{V}$ a $\boldsymbol{\mathcal{V}}_\wedge$-valued 2-form. Here, the symbol $\wedge$ is understood as antisymmetrization on both spacetime indices and internal Lorentz indices. Then,

$$\wedge : \Omega^1(\mathcal{X} \otimes \mathcal{V}) \times \Omega^1(\mathcal{X} \otimes \mathcal{V}) \to \Omega^2(\mathcal{X} \otimes \boldsymbol{\mathcal{V}}^q_\wedge)$$

As such, the object $\mathfrak{U}_\wedge = e \wedge e$ is written:

$$\mathfrak{U}_\wedge = \frac{1}{2!2!}(\mathfrak{U}_\wedge)^{IJ}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes \mathfrak{e}_I \wedge \mathfrak{e}_J \quad \text{with} \quad (\mathfrak{U}_\wedge)^{IJ}_{\mu\nu} = e^I_\alpha e^J_\beta \tag{1.78}$$

The related basis $\mathfrak{e}_I \wedge \mathfrak{e}_J = \mathfrak{e}_I \otimes \mathfrak{e}_J - \mathfrak{e}_J \otimes \mathfrak{e}_I$ of $\boldsymbol{\mathcal{V}}^2_\wedge = \Lambda^2\mathcal{V}$ leads us to decompose :

$$\mathfrak{U}_\wedge = (\frac{1}{2!2!})(\mathfrak{U}_\wedge)^{IJ}_{\mu\nu}\big[\mathfrak{e}_I \wedge \mathfrak{e}_J\big] \otimes \big[\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\big] = \frac{1}{4}(\mathfrak{U}_\wedge)^{IJ}_{\mu\nu}\mathfrak{e}_I \wedge \mathfrak{e}_J \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu.$$

with $(\mathfrak{U}_\wedge)^{IJ}_{\mu\nu} = e^I_\mu e^J_\nu$ So that, pictured as a tensor, we find:

$$\begin{aligned}
\mathfrak{U}_\wedge &= \frac{1}{4}\big[\mathfrak{U}^{IJ}_\wedge{}_{\mu\nu}\big]\big[\mathfrak{e}_I \otimes \mathfrak{e}_J - \mathfrak{e}_J \otimes \mathfrak{e}_I\big] \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
&= \frac{1}{4}\big[\mathfrak{U}^{IJ}_\wedge{}_{\mu\nu} - \mathfrak{U}^{JI}_\wedge{}_{\mu\nu}\big]\mathfrak{e}_I \otimes \mathfrak{e}_J \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
&= \frac{1}{2}\big[\mathfrak{U}^{[IJ]}_\wedge{}_{\mu\nu}\big]\mathfrak{e}_I \otimes \mathfrak{e}_J \otimes \big[\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu\big] \\
&= \frac{1}{2}\big[\mathfrak{U}^{[IJ]}_\wedge{}_{\mu\nu} - \mathfrak{U}^{[IJ]}_\wedge{}_{\nu\mu}\big]\mathfrak{e}_I \otimes \mathfrak{e}_J \otimes \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \\
&= \mathfrak{U}^{[IJ]}_\wedge{}_{[\mu\nu]}\mathfrak{e}_I \otimes \mathfrak{e}_J \otimes \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu
\end{aligned} \tag{1.79}$$

which is equivalently written, if decomposed as a tensor field:

$$\mathfrak{U}_\wedge = e^{[I}{}_{[\mu}e^{J]}_{\nu]}\mathfrak{e}_I \otimes \mathfrak{e}_J \otimes \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \tag{1.80}$$

Notice that we write $\mathfrak{U}^{IJ}_\wedge = (e \wedge e)^{IJ} = e^I \wedge e^J$ and that:

$$\mathfrak{U}_\wedge = (e \wedge e)^{IJ} \otimes \mathfrak{e}_I \wedge \mathfrak{e}_J = e^I \wedge e^J \otimes \mathfrak{e}_I \wedge \mathfrak{e}_J \tag{1.81}$$

with $\mathfrak{U}^{IJ}_\wedge = (e \wedge e)^{IJ} = \frac{1}{4}(\mathfrak{U}_\wedge)^{IJ}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{4}e^I_\mu e^J_\nu \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$.

1.5.4 Relation between $\mathfrak{U}_\wedge = e \wedge e$ and $\mathfrak{U}_\otimes = e\wedge e$

For $\boldsymbol{\varphi}^I \in \Lambda^k\mathcal{V}, \boldsymbol{\eta}^J \in \Lambda^p\mathcal{V}$ Notice that we have the relation between

$$\boldsymbol{\varphi}^I \wedge \boldsymbol{\eta}^J = \frac{(k+p)!}{k!p!}\mathsf{Alt}(\boldsymbol{\varphi}^I \otimes \boldsymbol{\eta}^J) \qquad \text{with} \qquad \mathsf{Alt}(\boldsymbol{\varphi}^I \otimes \boldsymbol{\eta}^J) = \frac{1}{k!}\sum_{\sigma \in \mathcal{S}_k}(-1)^\sigma(\boldsymbol{\varphi}^I \otimes \boldsymbol{\eta}^J) \circ \sigma$$

so that in our case of interest, with e^I, e^J:

$$\begin{aligned}
e \wedge e &= e^I \wedge e^J \otimes (\mathfrak{e}_I \wedge \mathfrak{e}_J) = e^I \wedge e^J \otimes \big[\frac{(2)!}{1!1!}\mathsf{Alt}(\mathfrak{e}_I \otimes \mathfrak{e}_J)\big] \\
&= e^J \wedge e^J\big[\mathfrak{e}_I \otimes \mathfrak{e}_J - \mathfrak{e}_J \otimes \mathfrak{e}_I\big] = \big[e^I \wedge e^J - e^J \wedge e^I\big]\mathfrak{e}_I \otimes \mathfrak{e}_J \\
&= 2e^I \wedge e^J \mathfrak{e}_I \otimes \mathfrak{e}_J = 2e \wedge e
\end{aligned} \tag{1.82}$$

1.5.5 Isomorphism between bivector and Lie algebra $\mathfrak{so}(4)$ or $\mathfrak{so}(1,3)$

Isomorphism between the space of bivectors - which are thought to be anti-symmetric tensors - and the Lie algebra $\mathfrak{so}(4)$ or $\mathfrak{so}(1,3)$. First we concentrate on the decomposition $\mathfrak{so}(4) = \mathfrak{su}(2) \otimes \mathfrak{su}(2)$. We consider the generators of the Lorentz group $\mathsf{SO}(1,3)$, which are denoted by $\boldsymbol{\Delta}^{\mathsf{SO}(1,3)}_{IJ} = \boldsymbol{\Delta}_{IJ}$. We associate to any bivector $v \in \boldsymbol{\mathcal{V}} = \mathcal{V} \wedge \mathcal{V}$ a Lie algebra element $\boldsymbol{\eta}_{\mathcal{V}}$ via an isomorphism $v \mapsto \boldsymbol{\eta}_{\mathcal{V}} \in \mathfrak{so}(1,3)$.

1.5.6 The case of the $\mathcal{V}$-valued 2-form $\mathfrak{U} = e \wedge e \in \Omega^2(\mathcal{X}) \otimes \boldsymbol{\mathcal{V}}$

From now, we denote $\mathfrak{U} = \mathfrak{U}_{\wedge} = e \wedge e$ in order to simplify the notations. Keeping trace of Minkowski indices I, J is equivalent to describe $\Lambda^2\mathcal{V}$-valued 2-form, or a bivector-valued 2-form as

$$\mathfrak{U} = \mathfrak{U}^{IJ} \otimes \mathfrak{e}_I \wedge \mathfrak{e}_J = (e \wedge e)^{IJ} \otimes \mathfrak{e}_I \wedge \mathfrak{e}_J \quad \text{with} \quad \mathfrak{U}^{IJ} = \frac{1}{2}\Big[\frac{1}{2!}\mathfrak{U}^{IJ}{}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\Big]$$

from (1.80), we find that:

$$\mathfrak{U} \;=\; e^{[I}_{[\mu}e^{J]}_{\nu]}\mathfrak{e}_I \otimes \mathfrak{e}_J \otimes \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \frac{1}{2}e^I_{[\mu}e^J_{\nu]}\mathfrak{e}_I \wedge \mathfrak{e}_J \otimes \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \frac{1}{2}e^{[I}_{\mu}e^{J]}_{\nu}\mathfrak{e}_I \otimes \mathfrak{e}_J \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{1.83}$$

Notice that we have always the possibility to apply the *external* Hodge operator on $\mathfrak{U}$: we directly apply the definition (1.2)

$$\star\mathfrak{U} = \frac{1}{2}\big[\star\mathfrak{U}\big]_{\rho\sigma}{}^{IJ}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \otimes \mathfrak{e}_I \wedge \mathfrak{e}_J \tag{1.84}$$

with $\big[\star\mathfrak{U}\big]^{IJ}_{\rho\sigma} = \frac{1}{2}\boldsymbol{\varepsilon}^{\mu\nu}{}_{\rho\sigma}e^I_\mu e^J_\nu$. Then, the term $\big[\star\mathfrak{U}\big]_{\rho\sigma}{}^{IJ}$ in (1.84) is expand as the following :

$$\big[\star\mathfrak{U}\big]^{IJ}_{\mu\nu} = \frac{1}{2}\boldsymbol{\varepsilon}^{\rho\sigma}{}_{\mu\nu}e^I_\rho e^J_\sigma = \frac{1}{2}\boldsymbol{\varepsilon}^{\rho\sigma\alpha\beta}g_{\alpha\mu}g_{\beta\nu}e^I_\rho e^J_\sigma = \frac{1}{2}\boldsymbol{\varepsilon}^{KLMN}e^\rho_K e^\sigma_L e^\alpha_M e^\beta_N g_{\alpha\mu}g_{\beta\nu}e^I_\rho e^J_\sigma \tag{1.85}$$

Then we expand in (1.85) the terms $g_{\alpha\mu} = \eta_{OP}e^O_\alpha e^P_\mu$ and $g_{\beta\nu} = \eta_{QR}e^Q_\beta e^R_\nu$ in order to obtain :

$$\begin{aligned} 2\big[\star\mathfrak{U}\big]^{IJ}_{\mu\nu} &= \varepsilon^{KLMN}e^\rho_K e^\sigma_L e^\alpha_M e^\beta_N\big[\eta_{OP}e^O_\alpha e^P_\mu\big]\big[\eta_{QR}e^Q_\beta e^R_\nu\big]e^I_\rho e^J_\sigma \\ &= \varepsilon^{KLMN}e^\rho_K e^\sigma_L e^P_\mu\big[\delta^O_M\eta_{OP}\big]\big[\delta^Q_N\eta_{QR}\big]e^R_\nu e^I_\rho e^J_\sigma \\ &= \varepsilon^{KLMN}e^\rho_K e^\sigma_L e^P_\mu\big[\delta^O_M\eta_{OP}\big]\big[\delta^Q_N\eta_{QR}\big]e^R_\nu e^I_\rho e^J_\sigma \\ &= \varepsilon^{KLOQ}\eta_{OP}\eta_{QR}\big[e^\rho_K e^\sigma_L e^I_\rho e^J_\sigma\big]e^P_\mu e^R_\nu \\ &= \varepsilon^{KL}{}_{PR}\delta^I_K\delta^J_L e^P_\mu e^R_\nu \\ &= \varepsilon^{IJ}{}_{PR}e^P_\mu e^R_\nu \end{aligned} \tag{1.86}$$

Now, we apply the *internal* Hodge operator $\star$ to $e \wedge e$, and once again we directly apply the definition 1.2 hence

$$\star\mathfrak{U} = \frac{1}{2}\big[\star\mathfrak{U}\big]^{KL}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes \mathfrak{e}_K \wedge \mathfrak{e}_L \qquad \text{with} \qquad \big[\star\mathfrak{U}\big]^{IJ}_{\mu\nu} = \frac{1}{2}\boldsymbol{\varepsilon}^{IJ}{}_{KL}e^K_\mu e^L_\nu \tag{1.87}$$

Therefore we obtain, from (1.86) and (1.87)(ii) : $\big[\star\mathfrak{U}\big]^{IJ}_{\mu\nu} = \big[\star\mathfrak{U}\big]^{IJ}_{\mu\nu}$ ⌋

Lemma 1.1. *With a Hodge operator either on internal indices* ☆ *or on space-time indices* $\star$ *defined above, the tetrad field, seen as a Minkowski vector valued* 1*-form and* $\mathfrak{U} = e \wedge e$ *- which is perceived as a* $\boldsymbol{\mathcal{V}}$*-valued* 2*-form. We have*

$$☆\mathfrak{U} = ☆(e \wedge e) = \star(e \wedge e) = \star\mathfrak{U}$$

We can choose to *not* write the Minkowski-like basis (namely $\mathfrak{e}_I \wedge \mathfrak{e}_J$) on $\Lambda^2\mathcal{V}$ indices so we write in more simple notation :

$$\star\mathfrak{U}^{KL} = \frac{1}{2}\boldsymbol{\varepsilon}^{\rho\sigma}{}_{\mu\nu} e^K_\rho e^L_\sigma \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \star\mathfrak{U}^{KL}$$

Therefore we observe the following notation :

$$\star\mathfrak{U}^{IJ} = \frac{1}{2}\varepsilon^{IJ}{}_{KL}\mathfrak{U}^{KL}{}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \qquad \text{and} \qquad \star\mathfrak{U}^{IJ} = \frac{1}{2}e\varepsilon^{\mu\nu}{}_{\rho\sigma}\mathfrak{U}^{IJ}{}_{\mu\nu}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma$$

It is then straightforward to find back, with $\Sigma_{IJ} = \star(e_I \wedge e_J)$:

$$\begin{aligned} \mathcal{L}_{\mathsf{Palatini}}[e,\omega] &= \int_{\mathcal{X}} \mathfrak{U}_{\alpha\beta} \wedge \mathbf{R}^{\alpha\beta} = \int_{\mathcal{X}} (e^\alpha_I e^\beta_J \mathfrak{U}_{\alpha\beta}) \wedge (e^I_\alpha e^J_\beta \mathbf{R}^{\alpha\beta}) = \int_{\mathcal{X}} \mathfrak{U}_{IJ} \wedge \mathbf{F}^{IJ} \\ &= \int_{\mathcal{X}} \star(e_I \wedge e_J) \wedge \mathbf{F}^{IJ} \end{aligned} \tag{1.88}$$

Let notice that we may alternatively work with the following action :

$$\mathcal{L}_{\mathsf{Palatini}}[e,\omega] = \int_{\mathcal{X}} \star(e_I \wedge e_J) \wedge \mathbf{F}^{IJ} = \int_{\mathcal{X}} \star(e_I \wedge e_J) \wedge \mathbf{F}^{IJ} \tag{1.89}$$

Notice that we have: $\mathbf{F}_{IJ} \wedge \star(e^I \wedge e^J) = \star\mathbf{F}_{IJ} \wedge (e^I \wedge e^J)$.

1.6 Einstein-Cartan Hodge and differential forms

We make some remarks on Einstein-Cartan fields equations, using differential forms. We have the following system of equations :

$$\left| \begin{array}{rcl} \varepsilon_{IJKL} e^J \wedge \mathbf{F}^{KL} &=& 0 \\ \mathrm{d}_\omega(e^K \wedge e^L) &=& 0 \end{array} \right. \tag{1.90}$$

With the help of Hodge duality, the equations of movement are straightforward derived. Let us compute the variation of the action. For the term $\delta_\omega\mathcal{L}_{\mathsf{Palatini}}[e,\omega]$ - the full treatment would be the following variation:

$$\delta\mathcal{L}_{\mathsf{Palatini}}[e,\omega] = \delta_e\mathcal{L}_{\mathsf{Palatini}}[e,\omega] + \delta_\omega\mathcal{L}_{\mathsf{Palatini}}[e,\omega] \tag{1.91}$$

we describe variation with respect to ω :

$$\delta_\omega\mathcal{L}_{\mathsf{Palatini}} = \delta_\omega\Big(\int_{\mathcal{X}} \star(e^I \wedge e^J) \wedge \mathbf{F}_{IJ}\Big) = \delta_\omega\Big(\int_{\mathcal{X}} \star\mathfrak{U}^{IJ} \wedge \mathbf{F}_{IJ}\Big) = \int_{\mathcal{X}} \star\mathfrak{U}^{IJ} \wedge \mathrm{d}_\omega\delta\omega_{IJ}$$

Since $\delta_\omega\mathbf{F}_{IJ} = \mathrm{d}\big(\delta_\omega(\omega_{IJ})\big) + [\omega,\delta\omega]_{IJ} = \mathrm{d}_\omega(\delta_\omega\omega_{IJ}) = \mathrm{d}_\omega(\delta\omega_{IJ})$ we have :

$$\delta_\omega\mathcal{L}_{\mathsf{Palatini}} = \frac{1}{2}\int \varepsilon^{IJ}{}_{KL}\mathfrak{U}^{KL} \wedge \mathrm{d}_\omega\delta\omega_{IJ} = -\frac{1}{2}\int \varepsilon^{IJ}{}_{KL}\mathrm{d}_\omega\mathfrak{U}^{KL} \wedge \delta\omega_{IJ}$$

Then, $\delta_\omega\mathcal{L}_{\mathsf{Palatini}} = 0 \iff \mathrm{d}_\omega\mathfrak{U}^{KL} = \mathrm{d}_\omega(e^K \wedge e^L) = 0$. Following Wise [57] we also notice the following equivalent formulation for Palatini gravity. We write the Palatini action as (1.92) :

$$\mathcal{L}_{\mathsf{Palatini}}[e,\omega] = \int \langle\, e \wedge e \wedge \mathbf{F} \,\rangle \tag{1.92}$$

The expression $e \wedge e \wedge \mathbf{F}$ is a $\Lambda^4\mathcal{V}$-valued 4-form on $\mathcal{X}$ (so that $e \wedge e \wedge \mathbf{F} \in \Omega^4(\mathcal{X}) \otimes \boldsymbol{\mathcal{V}}^4_\wedge$) while $\langle,\rangle$ is a trace - build on the the *internal* Hodge operator $\star$ -, which turns such a form into an ordinary real-valued 4-form. The wedge product $\wedge$ acts both on space-time indices and on internal Lorentz indices. **F** is the curvature of ω, described as a $\Lambda^2\mathcal{V}$-valued 2-form. $\langle , \rangle : \Omega(\mathcal{X}, \Lambda^4\mathcal{V}) \to \Omega(\mathcal{X}, \mathbb{R})$. Then performing variation with respect to ω and e, we obtain :

$$\delta\mathcal{L}_{\text{Palatini}} = \int \langle 2\delta e \wedge e \wedge \mathbf{F} + \mathrm{d}_\omega(e \wedge e) \wedge \delta\omega \rangle$$

The equations of motion are written by :

$$\left| \begin{array}{lcl} e \wedge \mathbf{F} & = & 0 \\ \mathrm{d}_\omega(e \wedge e) & = & 0 \end{array} \right. \tag{1.93}$$

Following D.K. Wise [57] we focus on the interplay of *internal* and *external* Hodge duality - the relation

$$\star \underbrace{\left[e \wedge \cdots \wedge e\right]}_{\text{p times}} = \frac{(n-p)!}{p!} \star \underbrace{\left[e \wedge \cdots \wedge e\right]}_{\text{n-p times}} \tag{1.94}$$

In $4D$ case we found $e \in \mathcal{V} \otimes T^\star\mathcal{X}$ and $e \wedge e \wedge e \in \Lambda^3\mathcal{V} \otimes \Lambda^3 T^\star\mathcal{X}$, so that $\star e$ and $\star(e \wedge e \wedge e)$ are in $\Lambda^3\mathcal{V} \otimes T^\star\mathcal{X}$. In components :

$$\left| \begin{array}{lcl} \star e & = & \Big[\star e\Big]_\mu^{IJK} \mathrm{d}x^\mu \otimes e_I \wedge e_J \wedge e_K \\ \star(e \wedge e \wedge e) & = & \Big[\star(e \wedge e \wedge e)\Big]_\mu^{IJK} \mathrm{d}x^\mu \otimes e_I \wedge e_J \wedge e_K \end{array} \right. \tag{1.95}$$

we are not enter into details and refer to the work of Wise [57] or Baez [6] [7] for the underlying relation with more general BF theory.

2 Self-Dual-Solutions of Einstein's Equation

The self-dual solutions of Einstein's equation is related to the work of Plebanski, Sen and Ashtekar. First we introduce the so-called Plebanski self-dual 2-forms. We refer to the work of Plebanski [36], Capovilla, Jacobson, Dell and Mason, [13] [14] or the introduction given by Krasnov [23]. Self-Dual-Solutions of Einstein's Equation are related to the fact, that curvature and Einstein equation can be given by the means of self-dual 2-forms. First, we introduce the following objects:

$$\mathbf{\Sigma}_\triangleleft^{IJ} = \frac{1}{2}\left[e^I \wedge e^J - \frac{i}{2}\boldsymbol{\varepsilon}^{IJ}{}_{KL} e^K \wedge e^L\right] \tag{2.1}$$

Notice that we equivalently write:

$$\mathbf{\Sigma}_\triangleleft^{IJ} = \frac{1}{2}\left[\mathfrak{U}^{IJ} - \frac{i}{2}\boldsymbol{\varepsilon}^{IJ}{}_{KL}\mathfrak{U}^{KL}\right] = \frac{1}{2}\left[\mathfrak{U}^{IJ} - i\star\mathfrak{U}^{IJ}\right] = \left[\mathfrak{U}^\triangleleft\right]^{IJ} \tag{2.2}$$

Notice that $\mathbf{\Sigma}^{IJ}$ is self dual in Lorentz indices $\star\mathbf{\Sigma}^{IJ} = i\mathbf{\Sigma}^{IJ}$ as well as under the exterior Hodge dual, when perceived as a 2-forms $\star\mathbf{\Sigma}^{IJ} = i\mathbf{\Sigma}^{IJ}$.[21] The anti-self-dual object is written:

$$\mathbf{\Sigma}_\triangleright^{IJ} = \frac{1}{2}\left[e^I \wedge e^J + \frac{i}{2}\boldsymbol{\varepsilon}^{IJ}{}_{KL} e^K \wedge e^L\right] = \frac{1}{2}\left[\mathfrak{U}^{IJ} + i\star\mathfrak{U}^{KL}\right] = \left[\mathfrak{U}^\triangleright\right]^{IJ} \tag{2.3}$$

[21] ⌈ **Proof** Let us expand the expression $\star\mathbf{\Sigma}_\triangleleft^{IJ}$, by definition:

$$\begin{array}{rcl} \star\mathbf{\Sigma}_\triangleleft^{IJ} & = & \star\Big[\frac{1}{2}\big[e^I \wedge e^J - \frac{i}{2}\boldsymbol{\varepsilon}^{IJ}{}_{KL} e^K \wedge e^L\big]\Big] = \frac{1}{2}\Big[\star\big[e^I \wedge e^J\big] - \frac{i}{2}\boldsymbol{\varepsilon}^{IJ}{}_{KL}\star\big[e^K \wedge e^L\big]\Big] \\ & = & \frac{1}{2}\Big[\frac{1}{2}\varepsilon^{IJ}{}_{KL} e^K \wedge e^L - \frac{i}{2}\boldsymbol{\varepsilon}^{IJ}{}_{KL}\star\big[e^K \wedge e^L\big]\Big] = \frac{1}{2}\Big[\frac{1}{2}\varepsilon^{IJ}{}_{KL} e^K \wedge e^L - \frac{i}{2}\boldsymbol{\varepsilon}^{IJ}{}_{KL}\big[\frac{1}{2}\varepsilon^{KL}{}_{MN} e^M \wedge e^N\big]\Big] \\ & = & \frac{1}{2}\Big[\frac{1}{2}\varepsilon^{IJ}{}_{KL} e^K \wedge e^L - \frac{i}{4}\boldsymbol{\varepsilon}^{IJ}{}_{KL}\varepsilon^{KL}{}_{MN} e^M \wedge e^N\Big] \end{array}$$

and we have: $\boldsymbol{\Sigma}^{IJ}_{\triangleright} = -i\boldsymbol{\Sigma}^{IJ}_{\triangleright}$ We need to notice that if we generally denote

$$\boldsymbol{\Sigma}^{IJ} = \frac{1}{2}\big[e^I \wedge e^J - \frac{i}{2}\boldsymbol{\varepsilon}^{IJ}{}_{KL}e^K \wedge e^L\big] \tag{2.4}$$

then, [22] in this case, we denote: $\boldsymbol{\Sigma}^{IJ}_{\blacktriangleright}$ since the two form $\boldsymbol{\Sigma}^{IJ}$ is self dual. Hence, for the form $\boldsymbol{\Sigma}^{IJ}$, we have the coincidence: $\left| \begin{array}{lcl} \boldsymbol{\Sigma}^{IJ}_{\blacktriangleleft} & = & \boldsymbol{\Sigma}^{IJ}_{\triangleleft} \\ \boldsymbol{\Sigma}^{IJ}_{\blacktriangleright} & = & \boldsymbol{\Sigma}^{IJ}_{\triangleright} \end{array} \right.$

2.1 Plebanski Gravity

We slightly modify the convention in the form (2.4), so that we define:

$$\boldsymbol{\Sigma}^{IJ} = ie^I \wedge e^J - \frac{1}{2}\boldsymbol{\varepsilon}^{IJ}{}_{KL}e^K \wedge e^L \tag{2.6}$$

From (2.7) we construct the following triple of forms $\boldsymbol{\Sigma}^i$ where $i \in 1, 2, 3$.

$$\boldsymbol{\Sigma}^i = ie^0 \wedge e^i - \frac{1}{2}\varepsilon^i{}_{jk}e^k \wedge e^k \tag{2.7}$$

written more explicitely:

$$\left| \begin{array}{lcl} \boldsymbol{\Sigma}^1 & = & ie^0 \wedge e^1 - (1/2)\varepsilon^i{}_{jk}e^k \wedge e^k \\ \boldsymbol{\Sigma}^2 & = & ie^0 \wedge e^2 - (1/2)\varepsilon^i{}_{jk}e^k \wedge e^k \\ \boldsymbol{\Sigma}^3 & = & ie^0 \wedge e^3 - (1/2)\varepsilon^i{}_{jk}e^k \wedge e^k \end{array} \right. \tag{2.8}$$

2.2 SL$(2, \mathbb{C})$-Plebanski formulation of Gravity

Here we follow various the presentations [36] [14] [49] In the original work of Plebanski, we find the use of a $\mathsf{SL}(2, \mathbb{C})$-spin connection, $\mathcal{A}^{AB}$ and a triple of self-dual 2-forms $\boldsymbol{\Sigma}^{AB}$, where: $\boldsymbol{\Sigma}^{AB} = \frac{1}{2}\theta^{AA'} \wedge \theta^B{}_{A'}$. The field $\theta^{AA'}$ is the tetrad field of the Cartan formalism (solder form), as a $\mathsf{SL}(2, \mathbb{C})$ object. So that:

$$\mathcal{L}_{\mathsf{Plebanski}}[\mathcal{A}, \boldsymbol{\Sigma}, \Psi] = \int \mathbf{F}_{AB} \wedge \boldsymbol{\Sigma}^{AB} - \frac{1}{2}\Psi_{ABCD}\boldsymbol{\Sigma}^{AB} \wedge \boldsymbol{\Sigma}^{CD} \tag{2.9}$$

Here, the curvature of the $\mathsf{SL}(2, \mathbb{C})$-connection $\mathcal{A}$ is denoted: $\mathbf{F}_{AB} = \mathrm{d}\mathcal{A}_{AB} - \mathbf{F}_{AC} \wedge \mathcal{A}_B{}^C$ and $\psi_{ABCD} = \psi_{(ABCD)}$ is totally symmetric. We find the following equation of motion, performing variation respectively to $\delta\Psi, \delta\mathcal{A}$ and $\delta\boldsymbol{\Sigma}$

$$\left| \begin{array}{rcl} 0 & = & \boldsymbol{\Sigma}^{(AB} \wedge \boldsymbol{\Sigma}^{CD)} \\ 0 & = & \mathrm{d}_{\mathcal{A}}\boldsymbol{\Sigma}^{AB} = \mathrm{d}\boldsymbol{\Sigma}^{AB} - 2\mathcal{A}^{(A}{}_C \wedge \boldsymbol{\Sigma}^{B)C} \\ \mathbf{F}_{AB} & = & \psi_{ABCD}\boldsymbol{\Sigma}^{CD} \end{array} \right. \tag{2.10}$$

with $\boldsymbol{\varepsilon}^{IJ}{}_{KL}\varepsilon^{KL}{}_{MN} = (-1)^{\boldsymbol{\sigma}}(2!)(2!)\delta^{[I}_M\delta^{J]}_N$. Then,

$$\begin{array}{rcl} \star\boldsymbol{\Sigma}^{IJ}_{\triangleleft} & = & \frac{1}{2}\Big[\frac{1}{2}\varepsilon^{IJ}{}_{KL}e^K \wedge e^L + i\delta^I_M\delta^J_N e^M \wedge e^N\Big] = \frac{i}{2}\Big[\frac{(-i)}{2}\varepsilon^{IJ}{}_{KL}e^K \wedge e^L + (-i)ie^I \wedge e^J\Big] \\ & = & \frac{i}{2}\Big[\frac{(-i)}{2}\varepsilon^{IJ}{}_{KL}e^K \wedge e^L + e^I \wedge e^J\Big] = i\Big[\frac{1}{2}\big[e^I \wedge e^J - \frac{i}{2}\varepsilon^{IJ}{}_{KL}e^K \wedge e^L\big]\Big] = i\boldsymbol{\Sigma}^{IJ}_{\triangleleft} \end{array}$$ ⌋

[22] ⌈ We have

$$\begin{array}{rcl} \star\boldsymbol{\Sigma}^{IJ} & = & \star\Big[\frac{1}{2}\big[\mathfrak{U}^{IJ} - i\star\mathfrak{U}^{IJ}\big]\Big] = \frac{1}{2}\big[\star\mathfrak{U}^{IJ} - i\star\star\mathfrak{U}^{IJ}\big] = \frac{1}{2}\big[\star\mathfrak{U}^{IJ} - i\star\star\mathfrak{U}^{IJ}\big] \\ & = & \frac{1}{2}\big[\star\mathfrak{U}^{IJ} + i\mathfrak{U}^{IJ}\big] = i\frac{(-i)}{2}\big[\star\mathfrak{U}^{IJ} + i\mathfrak{U}^{IJ}\big] = i\frac{1}{2}\big[\mathfrak{U}^{IJ} - i\star\mathfrak{U}^{IJ}\big] \\ & = & i\frac{1}{2}\big[\mathfrak{U}^{IJ} - i\star\mathfrak{U}^{IJ}\big] = i\boldsymbol{\Sigma}^{IJ} \end{array} \tag{2.5}$$

First, the relation $\mathbf{\Sigma}^{(AB} \wedge \mathbf{\Sigma}^{CD)}$ is equivalent that we can write $\mathbf{\Sigma}^{AB}$ under the form $\mathbf{\Sigma}^{AB} = \frac{1}{2}\theta^{AA'} \wedge \theta^{B}{}_{A'}$ The equation:

$$\mathrm{d}_{\mathcal{A}}\mathbf{\Sigma}^{AB} = \mathrm{d}\mathbf{\Sigma}^{AB} - 2\mathcal{A}^{(A}{}_{C} \wedge \mathbf{\Sigma}^{B)C} = 0 \tag{2.11}$$

is the translation that $\mathcal{A}^{AB}$ is identified with the the self dual part of the $\mathsf{SL}(2,\mathbb{C})$-connection $\mathcal{A}$. Recall that the torsion free spin connection in such a framework is denoted, $\omega^{AA'}{}_{BB'}$

2.3 Ashtekar-Sen-Gravity

Historically, A. Ashtekar [3] [4] [5] developed new variables for general relativity in relation with the work of A. Sen [41] [42] and we focus along this historical perspective on the so-called Ashtekar-Sen connection. The original Ashtekar connection $\mathbf{A}^i_a$ is the spatial projection of the self-dual part of the four dimensional spin connection.[23] We have the canonical variables[24] $(\mathbf{A}^i_a, \mathbf{E}^a_i)$ (2.12) :

$$\mathbf{A}^i_a = \Gamma^i_a + \gamma K^i_a = \frac{1}{2}\varepsilon^i{}_{jk}\omega^{jk}_a + \gamma\omega^{\circ i}_a \qquad \text{and} \qquad \mathbf{E}^a_i = \frac{1}{2}\varepsilon_{abc}\varepsilon^{ijk}e^b_j e^c_k \tag{2.12}$$

We denote equivalently $\mathbf{A}^i_a = 1/2\varepsilon^{ijk}\omega_{ajk} + \gamma K^i_a$. The usual Hamiltonian formalism process via the $(3+1)$-decomposition gives the set of constraints for Ashtekar phase space are the Gauss $\mathcal{G}_i$, vector $\mathcal{H}_a$ and scalar $\mathcal{H}$ constraints :

$$\left| \begin{array}{lcl} \mathcal{G}_i & = & \mathcal{D}_a\mathbf{E}^a_i \\ \mathcal{H} & = & (\mathbf{E})^{-1/2}\mathbf{E}^a_i\mathbf{E}^b_j\varepsilon^{ij}{}_k\mathbf{F}^k_{ab} \\ \mathcal{H}_a & = & \mathbf{E}^b_i\mathbf{F}^i_{ab} \end{array} \right. \tag{2.13}$$

Notice that if we consider the spin connection ω^{IJ} in the complexified bundle $\mathfrak{so}(1,3)\otimes\mathbb{C}$, we have denoted $\mathfrak{so}(1,3)_{\mathbb{C}} = so(1,3)\otimes\mathbb{C}$, the complexified $\mathfrak{so}(1,3)_{\mathbb{C}}$ lie algebra and decompose as

$$\mathfrak{so}(1,3)_{\mathbb{C}} = so(1,3)\otimes\mathbb{C} = \mathfrak{so}(1,3)^{\triangleleft}_{\mathbb{C}} \oplus \mathfrak{so}(1,3)^{\triangleright}_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \tag{2.14}$$

here, we do not here investigate spinor formalism with the group $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$ we prefer to use the group[25] $\mathfrak{su}(2)\otimes\mathbb{C}$, namely, we consider the decomposition:

$$\mathfrak{so}(1,3)_{\mathbb{C}} = \big[\mathfrak{su}(2)\otimes\mathbb{C}\big] \oplus \big[\mathfrak{su}(2)\otimes\mathbb{C}\big] = \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}} \tag{2.15}$$

Notice that $\omega \in \Omega^1(\mathcal{X}, \mathfrak{so}(1,3))$ has 6 algebraic independent components ω^{IJ}. Now if we consider the complexified connection $\omega_{\mathbb{C}} \in \Omega^1(\mathcal{X}, \mathfrak{so}(1,3)\otimes\mathbb{C}) = \Omega^1(\mathcal{X}, \mathfrak{so}(1,3)_{\mathbb{C}})$, then $\omega^{IJ}_{\mathbb{C}}$ has 6 complex algebraic independent components. The relation of self-duality or anti-self-duality reduce the number of independent components. Indeed, the complexified self-dual connection $\big[\omega^{\triangleleft}_{\mathbb{C}}\big]^{IJ}$ has only 3 independent components. We denote by $\mathfrak{a}(\boldsymbol{\varphi})^{IJ}$ the number of independent component so that we summarize the previous development. We have introduce the self-dual connection $\omega^{\triangleleft} = \mathfrak{P}^{\triangleleft}\omega = (1/2)\big[\omega - i{\star}\omega\big]$ or in components: $\big[\omega^{\triangleleft}\big]^{IJ} = \frac{1}{2}\big[\omega^{IJ} - i{\star}\omega^{IJ}\big]$. Now we consider the following variables: $\big[\mathbf{A}_{\mathsf{Ash}}\big]^i_{\mu} = \mathbf{A}^i_{\mu} = 2\big[\omega^{\triangleleft}\big]^{0i}_{\mu}$. What is important to notice is that the three

[23]In this section we denote $\boldsymbol{\sigma} = 1$ for Riemannian signature, and $\boldsymbol{\sigma} = -1$ for Euclidean one. Indices μ,ν,ρ denote space-time indices whereas $a,b,c...$ denote spatial indices and finaly $i,j,k...$ are related to $\mathfrak{su}(2)$ or $\mathfrak{so}(3)$ Lie algebra indices whereas, $I,J,K...$ denote $\mathfrak{so}(1,3)$ indices. In the literature concerning (LQG) one distinguish several pairs of canonical variables.

[24]Here, the parameter γ is equal to the imaginary number i

[25]We can also identify $\mathfrak{so}(3)\otimes\mathbb{C} = \mathfrak{su}(2)\otimes\mathbb{C} = \mathfrak{sl}(2,\mathbb{C})$. The decomposition at the Lie algebra level is $\mathfrak{so}(1,3)_{\mathbb{C}} = \big[\mathfrak{so}(3)\otimes\mathbb{C}\big] \oplus \big[\mathfrak{so}(3)\otimes\mathbb{C}\big] = \mathfrak{so}(3)_{\mathbb{C}} \oplus \mathfrak{so}(3)_{\mathbb{C}}$ which involves the group $\mathfrak{so}(3)\otimes\mathbb{C}$,

complex component $\mathbf{A}^1_\mu, \mathbf{A}^2_\mu$ and $\mathbf{A}^3_\mu$ describe all the independant components. More explicitly, due to self-duality we have constraints and there is only three independant components:

$$\left| \begin{array}{lll} \mathbf{A}^1_\mu & = & 2\big[\omega^{\triangleleft}\big]^{01}_\mu = \big[\omega^{01}_\mu - i\star\omega^{01}_\mu\big] \\ \mathbf{A}^2_\mu & = & 2\big[\omega^{\triangleleft}\big]^{02}_\mu = \big[\omega^{02}_\mu - i\star\omega^{02}_\mu\big] \\ \mathbf{A}^3_\mu & = & 2\big[\omega^{\triangleleft}\big]^{03}_\mu = \big[\omega^{03}_\mu - i\star\omega^{03}_\mu\big] \end{array} \right. \tag{2.16}$$

Notice that:

$$\star\omega^{IJ}_\mu = \frac{1}{2}\varepsilon^{IJ}{}_{KL}\omega^{KL}_\mu \quad \text{so that} \quad \big[\omega^{\triangleleft}\big]^{IJ}_\mu = \frac{1}{2}\Big[\omega^{IJ} - i\frac{1}{2}\varepsilon^{IJ}{}_{KL}\omega^{KL}_\mu\Big] \tag{2.17}$$

so that[26] the self dual part $\big[\omega^{\triangleleft}\big]^{IJ}_\mu$ verify: $\star\big[\omega^{\triangleleft}\big]^{IJ}_\mu = i\big[\omega^{\triangleleft}\big]^{IJ}_\mu$ so that when restricted to the three dimensional indices j, j, k we found:

$$\begin{array}{lll} \star\omega^{0i}_\mu & = & (1/2)\varepsilon^{0i}{}_{jk}\omega^{jk}_\mu \\ \star\omega^{jk}_\mu & = & (1/2)\varepsilon^{jk}{}_{0i}\omega^{0i}_\mu \end{array} \qquad \begin{array}{lll} \big[\omega^{\triangleleft}\big]^{0i}_\mu & = & (1/2)\Big[\omega^{0i} - i(1/2)\varepsilon^{0i}{}_{jk}\omega^{jk}_\mu\Big] \\ \big[\omega^{\triangleleft}\big]^{jk}_\mu & = & (1/2)\Big[\omega^{jk} - i(1/2)\varepsilon^{jk}{}_{0i}\omega^{0i}_\mu\Big] \end{array} \tag{2.20}$$

so that:

$$\left| \begin{array}{lll} \big[\omega^{\triangleleft}\big]^{0i}_\mu & = & \frac{1}{2}\Big[\omega^{0i} - i\frac{1}{2}\varepsilon^{0i}{}_{jk}\omega^{jk}_\mu\Big] = (-i)\frac{1}{2}\varepsilon^{0i}{}_{jk}\Big[\omega^{jk} - i\frac{1}{2}\varepsilon^{jk}{}_{0i}\omega^{0i}_\mu\Big] = -\frac{i}{2}\varepsilon^{0i}{}_{jk}\big[\omega^{\triangleleft}\big]^{jk}_\mu \\ \big[\omega^{\triangleleft}\big]^{jk}_\mu & = & \frac{1}{2}\Big[\omega^{jk} - i\frac{1}{2}\varepsilon^{jk}{}_{0i}\omega^{0i}_\mu\Big] = -i\varepsilon^{jk}{}_{0i}\big[\omega^{\triangleleft}\big]^{0i}_\mu \end{array} \right. \tag{2.21}$$

Notice that $\mathbf{A} = \mathbf{A}^i\boldsymbol{\sigma}_i \in \Omega^1(\mathcal{X}, \mathfrak{sl}(2,\mathbb{C}))$ with $\left| \begin{array}{lll} \mathbf{A}^1 & = & \big[\omega^{01} - i\star\omega^{01}\big]\boldsymbol{\sigma}_1 \\ \mathbf{A}^2 & = & \big[\omega^{02} - i\star\omega^{02}\big]\boldsymbol{\sigma}_2 \\ \mathbf{A}^3 & = & \big[\omega^{03} - i\star\omega^{03}\big]\boldsymbol{\sigma}_3 \end{array} \right.$ so that the Ashtekar-Sen variable is termed chiral, and the Ashtekar complex self/anti self dual connection is described by $\mathbf{A}_{\mathsf{Ash}} = \omega \pm i \star \omega$

3 BF Topological Field theory and Gravity

The BF theory is constructed on the following stones. First, we consider a principal G-bundle over $\mathcal{X}$, denoted $\mathcal{P}$. The Lie algebra of the group G is denoted $\mathfrak{g}$. Let ω be a connection 1-formsover the principle G-bundle, with $\dim(\mathcal{X}) = n$.[27] and let $\mathbf{B}$ be a tensorial $(n-2)$-form $\mathbf{B}$ of type ad. [28] The covariant derivative on $\Omega^*(\mathcal{X}, \mathrm{ad}\mathcal{P})$ and $\Omega^*(\mathcal{X}, \mathrm{ad}^\star\mathcal{P})$ is denoted d_ω. The

[26] ⌈ Proof

$$\begin{array}{lll} \star\big[\omega^{\triangleleft}\big]^{IJ}_\mu & = & \star\Big[\frac{1}{2}\Big[\omega^{IJ} - \frac{i}{2}\varepsilon^{IJ}{}_{KL}\omega^{KL}_\mu\Big]\Big] = \frac{1}{2}\Big[\star\big[e^I \wedge e^J\big] - \frac{i}{2}\varepsilon^{IJ}{}_{KL}\star\big[\omega^{KL}\big]\Big] \\ & = & \frac{1}{2}\Big[\frac{1}{2}\varepsilon^{IJ}{}_{KL}\omega^{KL} - \frac{i}{2}\varepsilon^{IJ}{}_{KL}\star\big[\omega^{KL}\big]\Big] = \frac{1}{2}\Big[\frac{1}{2}\varepsilon^{IJ}{}_{KL}\omega^{KL} - \frac{i}{2}\varepsilon^{IJ}{}_{KL}\Big[\frac{1}{2}\varepsilon^{KL}{}_{MN}\omega^{MN}\Big]\Big] \\ & = & \frac{1}{2}\Big[\frac{1}{2}\varepsilon^{IJ}{}_{KL}\omega^{KL} - \frac{i}{4}\varepsilon^{IJ}{}_{KL}\varepsilon^{KL}{}_{MN}\omega^{MN}\Big] \end{array} \tag{2.18}$$

with $\boldsymbol{\varepsilon}^{IJ}{}_{KL}\varepsilon^{KL}{}_{MN} = (-1)^{\boldsymbol{\sigma}}(2!)(2!)\delta^{[I}_M\delta^{J]}_N$. Then,

$$\begin{array}{lll} \star\big[\omega^{\triangleleft}\big]^{IJ}_\mu & = & \frac{1}{2}\Big[\frac{1}{2}\varepsilon^{IJ}{}_{KL}\omega^{KL} + i\delta^I_M\delta^J_N\omega^{MN}\Big] = \frac{i}{2}\Big[\frac{(-i)}{2}\varepsilon^{IJ}{}_{KL}\omega^{KL} + (-i)i\omega^{IJ}\Big] \\ & = & \frac{i}{2}\Big[\frac{(-i)}{2}\varepsilon^{IJ}{}_{KL}\omega^{KL} + \omega^{IJ}\Big] = i\Big[\frac{1}{2}\big[\omega^{IJ} - \frac{i}{2}\varepsilon^{IJ}{}_{KL}\omega^{KL}\big]\Big] \end{array} \tag{2.19}$$

So that: $\big[\omega^{\triangleleft}\big]^{IJ}_\mu = i\star\big[\omega^{\triangleleft}\big]^{IJ}_\mu$.

[27] We also denote by $\mathcal{A}$ the affine space of connection 1-forms.

[28] Notice that we describe by $\Omega^k(\mathcal{X}, \mathrm{ad}\mathcal{P})$ and $\Omega^k(\mathcal{X}, \mathrm{ad}^\star\mathcal{P})$ as the space of tensorial k-forms of type ad and ad*

curvature 2-form of ω, namely $\mathsf{F}_\omega \in \Omega^2(\mathcal{X}, \mathrm{ad}\mathcal{P})$. Notice that we consider the canonical paring between $\mathfrak{g}$ and $\mathfrak{g}^\star$ denoted $\langle , \rangle_{\mathfrak{g}}$. The Lagrangian density $\mathcal{L}_{\mathsf{BF}} = 1/2\langle \mathsf{B} \wedge \mathsf{F} \rangle$. Therefore the functional of interest is:

$$\mathcal{L}_{\mathsf{BF}}[\mathsf{B}, \mathsf{F}] = \frac{1}{2}\int_{\boldsymbol{\mathcal{X}}} \langle \mathsf{B} \wedge \mathsf{F} \rangle = \int_{\boldsymbol{\mathcal{X}}} \mathsf{B}^{IJ} \wedge \mathsf{F}_{IJ} \tag{3.1}$$

We observe the following the following two gauge symmetries. The gauge symmetry **I** and the Gauge symmetry **II** are respectively described by:

$$\begin{array}{lll} \omega \mapsto \mathsf{Ad}(g)\omega + g\mathrm{d}g^{-1} & \text{and} & \mathsf{B} \mapsto \mathsf{Ad}(g)\mathsf{B} \\ \omega \mapsto \omega & \text{and} & \mathsf{B} \mapsto \mathsf{B} + \mathrm{d}_\omega \boldsymbol{\eta} \end{array} \tag{3.2}$$

as well as the diffeomorphisms. The equation of movement (on-shell) are described by the objects $(\omega, \mathsf{B}) \in \mathcal{A} \times \Omega^{n-2}(\mathcal{X}, \mathrm{ad}^\star\mathcal{P})$ such that $\mathsf{F}_\omega = 0$. So that the connection ω is flat. Allso, $\mathrm{d}_\omega \mathsf{B} = 0$ the tensorial $(n-2)$-form is covariantly constant.

3.1 Traces and Killing forms

3.1.1 Lie group and Lie algebra Representation

Let consider a vectorial space $\mathcal{V}$ of finite dimension and let G be a Lie group. A representation ρ of G on the vectorial space $\mathcal{V}$ is a Lie group homomorphism: $\rho : G \to \mathsf{GL}(\mathcal{V})$ where $\mathsf{GL}(\mathcal{V})$ is the group of inversible endomorphisms on $\mathcal{V}$. A representation of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathsf{End}(\mathfrak{g})$. We define the adjoint action of $X \in \mathfrak{g}$ on $\mathfrak{g}$ as the endomorphism: $\mathsf{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ with $\mathsf{ad}(X)(Y) = [X, Y]$ for all $Y \in \mathfrak{g}$. Then, for any $X, Y \in \mathfrak{g}$, we have the adjoint representation defined by the map:

$$\begin{array}{rcl} \mathsf{ad} : \mathfrak{g} & \to & \mathsf{End}(\mathfrak{g}) \\ X & \mapsto & \mathsf{ad}(X) \end{array} \tag{3.3}$$

with for any $Y \in \mathfrak{g}$ $\mathsf{ad}(X) = [X, Y]$ which is the adjoint representation of $\mathfrak{g}$.

3.1.2 Bilinear form associated to a representation

We denote $\rho_{\mathcal{V}}$ a representation of $\mathfrak{g}$ on the vector space $\mathcal{V}$. Let us define the bilinear symmetric form on $\mathcal{V}$: $\beta : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$. Then, for any $v_1, v_2 \in \mathcal{V}$ $\beta(v_1, v_2) = \beta(v_2, v_1)$, and β is invariant with respect to $\rho_{\mathcal{V}}$ if for any $X \in \mathfrak{g}$ and for all $v_1, v_2 \in \mathcal{V}$

$$\beta(\rho_{\mathcal{V}}(X)v_1, v_2) + \beta(v_1, \rho_{\mathcal{V}}(X)v_2) = 0 \tag{3.4}$$

on the other side, we define the bilinear symmetric form associate with the representation ρ as the following object:

$$\beta_\rho : \left\{ \begin{array}{rcl} \mathfrak{g} \times \mathfrak{g} & \to & \mathbb{R} \\ (X, Y) & \mapsto & \mathsf{tr}(\rho(X)\rho(Y)) \end{array} \right. \tag{3.5}$$

so that $\beta_\rho(X, Y) = \mathsf{tr}(\rho(X)\rho(Y))$. An important case is the one of Killing form, which is perceived as an inner product on the Lie algebra.

3.1.3 Killing form

The Killing form on the Lie algebra $\mathfrak{g}$ is the bilinear form

$$\kappa(X, Y) = \mathsf{Tr}\big(\mathsf{ad}(X) \circ \mathsf{ad}(Y)\big) \tag{3.6}$$

The Lie algebra $\mathfrak{g}$ is semisimple if the Killing form $\kappa(\cdot,\cdot)$ is non degenerate. A Lie algebra $\mathfrak{g}$ is simple if it is semisimple and if $\mathfrak{g}$ has no non-trivial ideals. Recall that a an ideal of $\mathfrak{g}$ is a subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. The Killing form $\kappa(\cdot,\cdot)$ is the scalar product of the Lie algebra $\mathfrak{g}$ defined in terms of the adjoint representation. The Killing form is described as a symmetric bilinear form κ, we denote $\kappa \in (\mathfrak{g}^\star)^{\odot 2}$ or $\kappa \in \bigvee^2(\mathfrak{g}^\star)$. Notice that we equivalently write the symmetric bilinear forms with for any $X, Y, Z \in \mathfrak{g}$

$$\kappa(X,Y) = \kappa(Y,X) \qquad \text{and} \qquad \kappa(X,[Z,Y]) + \kappa([Z,X],Y) = 0 \tag{3.7}$$

We construct the Killing form, from the adjoint representation ad.

3.1.4 The Yang-Mills functional

The Yang-Mills functional is given by: $\mathcal{L}_{\mathsf{YM}} = -\int_{\mathcal{X}} \mathsf{tr}(\mathbf{F} \wedge \star\mathbf{F})$. This is the first example of Lagrangian functional that is constructed with a trace over the Lie algebra. We consider here the $\mathfrak{su}(n)$ Lie algebra, so that the curvature 2-form is locally a $\mathfrak{su}(n)$-valued 2-form: $\mathbf{F} = \mathbf{F}^a_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes \boldsymbol{\Delta}_a$, where $\boldsymbol{\Delta}_a = \boldsymbol{\Delta}_a^{\mathfrak{su}(n)}$ are the generators of $\mathfrak{su}(n)$. Notice that we denote more simply: $\mathbf{F}_{\mu\nu} = \mathbf{F}^a_{\mu\nu}\boldsymbol{\Delta}_a$ such an object.[29] If we consider a representation of the Lie algebra $\mathfrak{su}(n)$, the curvature 2-form is written with matrix indices:

$$\big(\mathbf{F}_{\mu\nu}\big)_{ij} = \mathbf{F}^a_{\mu\nu}\big(\boldsymbol{\Delta}_a\big)_{ij}$$

We consider the inner product We write the Hodge star $\star\mathbf{F}$ in components : $\big(\star\mathbf{F}\big)_{\rho\sigma} = \frac{1}{2!}\boldsymbol{\varepsilon}^{\mu\nu}{}_{\rho\sigma}\mathbf{F}_{\mu\nu}$:

$$\begin{aligned} \star\mathbf{F} &= \star\big(\frac{1}{2!}\mathbf{F}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\big) = \frac{1}{2!}\big(\star\mathbf{F}\big)_{\rho\sigma}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma = \frac{1}{2!}\frac{1}{2!}\boldsymbol{\varepsilon}^{\mu\nu}{}_{\rho\sigma}\mathbf{F}_{\mu\nu}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \\ &= \frac{\sqrt{-g}}{4} g_{\alpha\mu}g_{\beta\nu}\varepsilon^{\mu\nu}{}_{\rho\sigma}\mathbf{F}^{\alpha\beta}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma = \frac{\sqrt{-g}}{4}\varepsilon^{\mu\nu}{}_{\rho\sigma}\mathbf{F}_{\mu\nu}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma. \end{aligned}$$

We have the following[30]

$$\mathbf{F} \wedge \star\mathbf{F} = -\frac{1}{2}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\sqrt{-g}\,\mathrm{d}\mathfrak{y}. \tag{3.8}$$

We fix the setting in the flat Minkowski space so that: $\mathbf{F} \wedge \star\mathbf{F} = -\frac{1}{2}\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\mathrm{d}\mathfrak{y}$. Now we are taking into account the trace of $\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}$, namely we consider the object $\mathsf{tr}(\mathbf{F} \wedge \star\mathbf{F})$. Hence, we consider

$$\mathsf{tr}(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}) = \mathsf{tr}(\mathbf{F}^a_{\mu\nu}\boldsymbol{\Delta}_a\mathbf{F}^{\mu\nu b}\boldsymbol{\Delta}_b) = \mathsf{tr}(\boldsymbol{\Delta}_a\boldsymbol{\Delta}_b)\mathbf{F}^a_{\mu\nu}\mathbf{F}^{\mu\nu b} = \frac{1}{2}\delta_{ab}\mathbf{F}^a_{\mu\nu}\mathbf{F}^{\mu\nu b} = \frac{1}{2}\mathbf{F}^a_{\mu\nu}\mathbf{F}^{\mu\nu a}$$

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[29] Notice that $\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu - i[\mathbf{A}_\mu, \mathbf{A}_\nu]$. The generators of the $\mathfrak{su}(n)$ Lie algebra are Hermitian.

[30] ⌈ **Proof** We make a straightforward computation which involves the Hodge duality.

$$\begin{aligned} \mathbf{F} \wedge \star\mathbf{F} &= \Big[\frac{1}{2}\mathbf{F}_{\lambda\varsigma}\mathrm{d}x^\lambda \wedge \mathrm{d}x^\varsigma\Big] \wedge \Big[\frac{\sqrt{-g}}{4}g_{\alpha\mu}g_{\beta\nu}\varepsilon^{\mu\nu}{}_{\rho\sigma}\mathbf{F}^{\alpha\beta}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma\Big] \\ &= \frac{1}{8}\Big[\mathbf{F}_{\lambda\varsigma}\sqrt{g}g_{\alpha\mu}g_{\beta\nu}\varepsilon^{\mu\nu}{}_{\rho\sigma}\mathbf{F}^{\alpha\beta}\Big]\mathrm{d}x^\lambda \wedge \mathrm{d}x^\varsigma \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma. \end{aligned}$$

Since $\mathrm{vol}_{\mathcal{X}}(g) = \sqrt{g}\mathrm{d}\mathfrak{y} = \frac{1}{4!}\boldsymbol{\varepsilon}_{\lambda\varsigma\rho\sigma}\mathrm{d}x^\lambda \wedge \mathrm{d}x^\varsigma \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma$, we obtain :

$$\begin{aligned} \mathbf{F} \wedge \star\mathbf{F} &= \frac{1}{8}\Big[\mathbf{F}_{\lambda\varsigma}\sqrt{g}g_{\alpha\mu}g_{\beta\nu}\varepsilon^{\mu\nu}{}_{\rho\sigma}\mathbf{F}^{\alpha\beta}\Big]\mathrm{d}x^\lambda \wedge \mathrm{d}x^\varsigma \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma = \frac{1}{8}\mathbf{F}_{\lambda\varsigma}\mathbf{F}^{\alpha\beta}\boldsymbol{\varepsilon}_{\alpha\beta\rho\sigma}\boldsymbol{\varepsilon}^{\lambda\varsigma\rho\sigma}\sqrt{g}\mathrm{d}\mathfrak{y} \\ &= -\frac{1}{2}\delta^{[\lambda}_\alpha\delta^{\varsigma]}_\beta\mathbf{F}_{\lambda\varsigma}\mathbf{F}^{\alpha\beta}\sqrt{g}\mathrm{d}\mathfrak{y} = -\frac{1}{2}\frac{1}{2}\Big[\mathbf{F}_{\alpha\beta}\mathbf{F}^{\alpha\beta} - \mathbf{F}_{\beta\alpha}\mathbf{F}^{\alpha\beta}\Big]\sqrt{g}\mathrm{d}\mathfrak{y} = -\frac{1}{2}\mathbf{F}_{\alpha\beta}\mathbf{F}^{\alpha\beta}\sqrt{g}\mathrm{d}\mathfrak{y} \end{aligned}$$

where we have used the identity $\boldsymbol{\varepsilon}_{\alpha\beta\rho\sigma}\boldsymbol{\varepsilon}^{\lambda\varsigma\rho\sigma} = -2!2!\delta^{[\lambda}_\alpha\delta^{\varsigma]}_\beta$ in the last line. ⌋

In the last line, we use: $\mathsf{tr}(\mathbf{\Delta}_a\mathbf{\Delta}_b) = \frac{1}{2}\delta_{ab}$. Then,

$$\mathcal{L}_{\mathsf{YM}} = -\int_{\mathcal{X}} \mathsf{tr}(\mathbf{F} \wedge \star\mathbf{F}) = -\frac{1}{4}\int_{\mathcal{X}} \mathbf{F}^a_{\mu\nu}\mathbf{F}^{\mu\nu a}\mathrm{d}\mathfrak{y}. \tag{3.9}$$

3.1.5 BF Theory

The basics field are the **B** filed, which is an $\mathsf{ad}(P)$-valued $(n-2)$-form on $\mathcal{X}$ and the curvature $\mathbf{F} = \mathrm{d}\omega + \omega \wedge \omega$ of ω, an $\mathsf{ad}(P)$-valued 2-form on $\mathcal{X}$. First we need to look at those objects locally. We choose a local trivialisation and we see the form **B** as an $\mathfrak{g}$-valued $(n-2)$-form on $\mathcal{X}$ and **F** as an $\mathfrak{g}$-valued 2-form on $\mathcal{X}$. This step allows us to write these objects locally: $\mathbf{B} = \mathbf{B}^i_{\mu_1,\cdots,\mu_{n-2}}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{n-2}} \otimes \mathfrak{b}_i$ Where $\{\mathfrak{b}_i\}_{1\leq i\leq d}$ is a basis of the lie algebra $\mathfrak{g}$, with $\mathsf{dim}(\mathfrak{g}) = d$ Also we write: $\mathbf{F} = \mathbf{F}^i_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes \mathfrak{b}_i$. The Lagrangian density, which is written

$$\boldsymbol{\mathcal{L}} = \mathsf{tr}(\mathbf{B} \wedge \mathbf{F})$$

is a n-form: $\boldsymbol{\mathcal{L}} \in \Omega^n(\mathcal{X})$. The operation involved is the following. On one hand, we take the wedge product for the differential form parts of each form involved, namely **B** and **F** whereas on the other hand, we use the natural paring on the Lie algebra $\langle,\rangle_{\mathfrak{g}}$

$$\mathsf{tr}(\mathbf{B} \wedge \mathbf{F}) = \mathbf{B}^i_{\mu_1,\cdots,\nu_{n-2}}\mathbf{F}^j_{\mu\nu}\langle\mathfrak{b}_i,\mathfrak{b}_j\rangle_{\mathfrak{g}}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_2} \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{3.10}$$

3.1.6 Example, tr$(e \wedge e \wedge \mathbf{F})$ and $\langle e \wedge e \wedge \mathbf{F}\rangle$

We define the fiber bundle P, as the oriented frame bundle of $\mathcal{X}$, a principal G-bundle with G being $\mathsf{SO}(4)$ - or $G = \mathsf{SO}(1,3)$. Then we consider the adjoint bundle $\mathsf{ad}(P)$. Recall that the adjoint bundle $\mathsf{ad}(P)$ is isomorphic to $\Lambda^2\mathcal{V}$. Hence the important isomorphism in that case is:

$$\Lambda^2\mathbb{R}^{1,3} \cong \mathfrak{so}(1,3) \quad \text{Lorentzian} \qquad \text{or} \qquad \Lambda^2\mathbb{R}^4 \cong \mathfrak{so}(4) \quad \text{Euclidean}$$

The case of $\mathsf{tr}(e \wedge e \wedge \mathbf{F})$. Here we consider the two ways to obtain the Lagrangian density, $\mathcal{L} = \mathsf{tr}(\mathbf{B}\wedge\mathbf{F})$ with $\mathbf{B} := e\wedge e$. The first way is to consider both the forms $e\wedge e \in \Omega^2(\mathcal{X})\otimes\mathfrak{so}(1,3)$ and $\mathbf{F} \in \Omega^2(\mathcal{X}) \otimes \mathfrak{so}(1,3)$ so that we recover the previous case of application (3.10) There is another equivalent way to describe this object. The expression $e \wedge e \wedge \mathbf{F}$ is a $\Lambda^4\mathcal{V}$-valued 4-form on $\mathcal{X}$ (so that $e \wedge e \wedge \mathbf{F} \in \Omega^4(\mathcal{X}) \otimes \boldsymbol{\mathcal{V}}^4_\wedge$) while $\langle,\rangle$ is a trace - build on the the *internal* Hodge operator ⋆ -, which turns such a form into an ordinary real-valued 4-form, in that case, we write $\mathcal{L} = \langle e \wedge e \wedge \mathbf{F}\rangle$

3.2 $SO(4)$-Plebanski action

The Plebanski gravity, is seen as a BF-constrained theory. The new constraint are quadratic constraints on the **B** $\mathfrak{so}(1,3)$-valued 2-forms. It comes with the original Plebanski functional:

$$\mathcal{L}_{\mathsf{Plebanski}}[\omega,\mathbf{B}] = \int_{\mathcal{X}} \mathbf{B}^{IJ} \wedge \mathbf{F}_{IJ}(\omega) - \frac{\Lambda}{4}\varepsilon_{IJKL}\mathbf{B}^{IJ} \wedge \mathbf{B}^{KL} \tag{3.11}$$

Plebanski gravity concerns the description of gravity in terms of bivectors and the geometry $\mathfrak{so}(4)$ valued bivectors in the Euclidien case and the $\mathfrak{so}(1,3)$-valued bivector in the Lorentzian case. This section explicitly follows the work of De Pietri and Freidel [16], De Pietri, Freidel, Krasnov, Puzio [17] or Livine, [24]. The $\mathfrak{so}(4)$ Plebanski action, is written with an $\mathfrak{so}(4)$ connection $\omega = \omega^{IJ}_\mu\mathbf{\Delta}_{IJ}\mathrm{d}x^\mu$ and a $\mathfrak{so}(4)$-valued two form:

$$\mathbf{B} = \frac{1}{2}\mathbf{B}^{IJ}_{\mu\nu}\mathbf{\Delta}_{IJ}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2}\mathbf{B}^{IJ}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes \mathbf{\Delta}_{IJ} \tag{3.12}$$

Notice that $\star\boldsymbol{\eta} = \frac{1}{2!}(\star\boldsymbol{\eta})^{KL}_{\mu_1\cdots\mu_p}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}\otimes\mathfrak{e}_J\wedge\mathfrak{e}_K$ with $(\star\boldsymbol{\eta})^{KL}_{\mu_1\cdots\mu_p} = \frac{1}{2}\varepsilon^{KL}{}_{IJ}(\boldsymbol{\eta})^{KL}_{\mu\nu}$ so that the internal Hodge dual of the $\mathfrak{so}(4)$-valued 2-form $\mathbf{B}$, denoted $\star\mathbf{B}$ is written:

$$\star\mathbf{B} = \frac{1}{2}(\star\mathbf{B})^{KL}_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\otimes\boldsymbol{\Delta}_{IJ} \qquad \text{with} \qquad (\star\mathbf{B})^{KL}_{\mu\nu} = \frac{1}{2}\varepsilon^{KL}{}_{IJ}\mathbf{B}^{KL}_{\mu\nu} \tag{3.13}$$

and a scalar symmetric matrix $\varphi_{[IJ][KL]} = \varphi_{IJKL}$ is symmetric with respect to the exchange of the pairs $[IJ]$ and $[KL]$ and is antisymmetric in the first and second pair: $\varphi_{IJKL} = -\varphi_{JIKL} = -\varphi_{IJLK}$[31].

$$\mathcal{L}_{\text{Plebanski}}[\omega,\mathbf{B},\boldsymbol{\varphi}] = \int_{\mathcal{X}}\mathbf{B}^{IJ}\wedge\mathbf{F}_{IJ}(\omega) - \frac{\Lambda}{4}\varepsilon_{IJKL}\mathbf{B}^{IJ}\wedge\mathbf{B}^{KL} - \frac{1}{2}\varphi(\mathbf{B})_{IJ}\wedge\mathbf{B}^{KL} \tag{3.14}$$

Notice that $\varphi(\mathbf{B})_{IJ} = \varphi_{IJKL}\mathbf{B}^{KL}$ so that:

$$\mathcal{L}_{\text{Plebanski}}[\omega,\mathbf{B},\boldsymbol{\varphi}] = \int_{\mathcal{X}}\mathbf{B}^{IJ}\wedge\mathbf{F}_{IJ}(\omega) - \frac{\Lambda}{4}\varepsilon_{IJKL}\mathbf{B}^{IJ}\wedge\mathbf{B}^{KL} - \frac{1}{2}\varphi_{IJKL}\mathbf{B}^{IJ}\wedge\mathbf{B}^{KL} \tag{3.15}$$

The action functional (3.15) is called the *non-chiral* Plebanski BF-type action when $\mathfrak{g} = \mathfrak{so}(4)$ or $\mathfrak{g} = \mathfrak{so}(1,3)$. Notice that if we consider the case where $\mathfrak{g} = \mathfrak{su}(2)$, the action functional (3.15) is called the *chiral* Plebanski BF-type action. If we respectively consider the variations $\frac{\delta\mathcal{L}_{\text{Plebanski}}}{\delta\omega^{IJ}_\mu}$, $\frac{\delta\mathcal{L}_{\text{Plebanski}}}{\delta\mathbf{B}^{IJ}_{\mu\nu}}$ and $\frac{\delta\mathcal{L}_{\text{Plebanski}}}{\delta\varphi_{IJKL}}$ The equation of motion are:

$$\left|\begin{array}{rcl} \mathrm{d}_\omega\mathbf{B} & = & \mathrm{d}\mathbf{B} + [\omega,\mathbf{B}] = 0 \\ \mathbf{F}^{IJ}(\omega) & = & \varphi^{IJKL}\mathbf{B}_{KL} \\ \mathbf{B}^{IJ}\wedge\mathbf{B}^{KL} & = & (-1)^{\boldsymbol{\sigma}}e\varepsilon^{IJKL} \end{array}\right. \tag{3.16}$$

From this perspective, we can decompose φ along the decomposition $(2,0)\oplus(0,2)\oplus(1,1)\oplus(0,0)$ which corresponds to decompose the four Lorentz indices object φ_{IJKL} with taking into account the desired symmetry, antisymmetry and traceless conditions. Hence, we denote:

$$\varphi_{IJKL} = \underset{(2,0)}{\boldsymbol{\Pi}}{}_{IJKL}^{\ ij}\,\varphi^{\triangleleft}_{ij} + \underset{(0,2)}{\boldsymbol{\Pi}}{}_{IJKL}^{\ ij}\,\varphi^{\triangleright}_{ij} + \underset{(1,1)}{\boldsymbol{\Pi}}{}_{IJKL}^{\ ij}\,\psi_{ij} + \underset{(0,0)}{\boldsymbol{\Pi}}{}_{IJKL}\,\psi_\circ \tag{3.17}$$

in fact, φ^{IJKL} is decomposed in irreductible components with respect to $\mathfrak{so}(4)$

$$\left|\begin{array}{ll} \varphi^{\triangleleft}_{ij} & \text{5 components. Left part of the Weyl} \\ \varphi^{\triangleright}_{ij} & \text{5 components. Right part of the Weyl} \\ \psi_{ij} & \text{9 components. Traceless part of the Ricci} \\ \psi_\circ & \text{1 components. Scalar curvature} \end{array}\right. \tag{3.18}$$

These algebraic considerations leads us to decompose the BF action (3.15) (with zero cosmological constant) as:

$$\mathcal{L}_{\text{Plebanski}}[\omega,\mathbf{B},\boldsymbol{\varphi}] = \mathcal{L}^{\triangleleft}_{\text{Plebanski}} + \mathcal{L}^{\triangleright}_{\text{Plebanski}} + \mathcal{L}^{\circ}_{\text{Plebanski}} \tag{3.19}$$

[31] Notice that the object φ_{IJKL} has $4^4 = 256$ components. The imposition of the symmetry $\varphi_{IJKL} = -\varphi_{JIKL} = -\varphi_{IJLK}$ reduces to 36 the number of independent components. Then, the imposition of the symmetry $\varphi_{IJKL} = \varphi_{KLIJ}$ reduces to 21 the number of independent components. Finally the traceless condition $\varepsilon^{IJKL}\varphi_{IJKL} = 0$ reduces the number of independent components to 20.

with (we follow explicitly [16]):

$$\left| \begin{array}{lcl} \mathcal{L}^{\triangleleft}_{\text{Plebanski}} & = & \int_{\mathcal{X}} \delta_{ij} (\mathbf{B}^{\triangleleft})^i \wedge (\mathbf{F}^{\triangleleft})^j - \frac{\psi_\circ}{2} \delta_{ij} (\mathbf{B}^{\triangleleft})^i \wedge (\mathbf{B}^{\triangleleft})^j - \frac{1}{2} \varphi^{\triangleleft}_{ij} (\mathbf{B}^{\triangleleft})^i \wedge (\mathbf{B}^{\triangleleft})^j \\ \mathcal{L}^{\triangleright}_{\text{Plebanski}} & = & \int_{\mathcal{X}} \delta_{ij} (\mathbf{B}^{\triangleright})^i \wedge (\mathbf{F}^{\triangleright})^j - \frac{\psi_\circ}{2} \delta_{ij} (\mathbf{B}^{\triangleright})^i \wedge (\mathbf{B}^{\triangleright})^j - \frac{1}{2} \varphi^{\triangleright}_{ij} (\mathbf{B}^{\triangleright})^i \wedge (\mathbf{B}^{\triangleright})^j \\ \mathcal{L}^{\circ}_{\text{Plebanski}} & = & - \int_{\mathcal{X}} \psi_{ij} (\mathbf{B}^{\triangleright})^i \wedge (\mathbf{B}^{\triangleleft})^j \end{array} \right. \tag{3.20}$$

In order to obtain the expression (3.20) we have decomposed the 2-form $\mathbf{B}$ and the connection 1-form into their self-dual and anti-self-dual parts:

$$\mathbf{B} = \mathbf{B}^{\triangleleft} + \mathbf{B}^{\triangleright} \qquad \text{and} \qquad \omega = \omega^{\triangleleft} + \omega^{\triangleright}$$

and the object φ is given as a sum following the algebraic decomposition (3.17) into irreductible components with respect to the Lie algebra $\mathfrak{so}(4)$. So that we write:

$$\varphi = \varphi^{\triangleleft}_{ij} + \varphi^{\triangleright}_{ij} + \psi_{ij} + \psi_\circ \tag{3.21}$$

As shown in [16], this decomposition leads to the following equations of movement:

$$\left| \begin{array}{ll} \frac{\delta \mathcal{L}_{\text{Plebanski}}}{\delta \varphi^{\triangleleft}_{ij}} & (\mathbf{B}^{\triangleleft})^i \wedge (\mathbf{B}^{\triangleleft})^j - \frac{1}{3} \delta^{ij} \delta_{kl} (\mathbf{B}^{\triangleleft})^k \wedge (\mathbf{B}^{\triangleleft})^l = 0 \\ \frac{\delta \mathcal{L}_{\text{Plebanski}}}{\delta \varphi^{\triangleright}_{ij}} & (\mathbf{B}^{\triangleright})^i \wedge (\mathbf{B}^{\triangleright})^j - \frac{1}{3} \delta^{ij} \delta_{kl} (\mathbf{B}^{\triangleright})^k \wedge (\mathbf{B}^{\triangleright})^l = 0 \\ \frac{\delta \mathcal{L}_{\text{Plebanski}}}{\delta \psi_{ij}} & (\mathbf{B}^{\triangleright})^i \wedge (\mathbf{B}^{\triangleleft})^j = 0 \\ \frac{\delta \mathcal{L}_{\text{Plebanski}}}{\delta \psi_\circ} & \delta_{ij} \big((\mathbf{B}^{\triangleleft})^i \wedge (\mathbf{B}^{\triangleleft})^j + (\mathbf{B}^{\triangleright})^i \wedge (\mathbf{B}^{\triangleright})^j \big) = 0 \end{array} \right. \tag{3.22}$$

We now more explicitly describe the decomposition (3.17). In particular we gives the expression of the objects[32] $\underset{(2,0)}{\mathbf{\Pi}}{}^{ij}_{IJKL}$, $\underset{(0,2)}{\mathbf{\Pi}}{}^{ij}_{IJKL}$, $\underset{(1,1)}{\mathbf{\Pi}}{}^{ij}_{IJKL} \psi_{ij}$ and $\underset{(0,0)}{\mathbf{\Pi}}{}_{IJKL} \psi_\circ$. For that purpose, we need to choose a good basis adapted to the (anti)-self-dual decomposition. First, recall that a basis of the $\mathfrak{so}(4)$ Lie algebra[33] is given via the basis $\{\mathsf{L}_i, \mathsf{K}_i\}$ where L_i are the so-called rotation generators whereas K_i are the boost generators, for $1 \leq i \leq 3$. In this case, the Lie algebra is describe via the following commutation relations:

$$\left| \begin{array}{lcl} [\mathsf{L}_i, \mathsf{L}_j] & = & \varepsilon_{ij}{}^k \mathsf{L}_k \\ [\mathsf{K}_i, \mathsf{K}_j] & = & \varepsilon_{ij}{}^k \mathsf{L}_k \\ [\mathsf{K}_i, \mathsf{L}_j] & = & \varepsilon_{ij}{}^k \mathsf{K}_k \end{array} \right. \tag{3.23}$$

Notice that, following [30], the infinitesimal generators in the 4-dimensional representation are given by the following matrices:

$$\mathsf{L}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathsf{L}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathsf{L}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathsf{K}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \mathsf{K}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \mathsf{K}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

[32] respectively as basis of each irreductible part $(2,0)$, $(0,2)$, $(1,1)$ and $(0,0)$

[33] the real Lie algebra of the isometry group $\mathsf{SO}(4)$

In the case where we consider the $\mathfrak{so}(1,3)$ Lie algebrawe have the following commutation relations:

$$\left| \begin{array}{rcl} [\mathsf{L}_i, \mathsf{L}_j] & = & \varepsilon_{ij}{}^k \mathsf{L}_k \\ [\mathsf{K}_i, \mathsf{K}_j] & = & -\varepsilon_{ij}{}^k \mathsf{L}_k \\ [\mathsf{K}_i, \mathsf{L}_j] & = & \varepsilon_{ij}{}^k \mathsf{K}_k \end{array} \right. \tag{3.24}$$

The infinitesimal generators in the 4-dimensional representation are given by the following matrices:

$$\mathsf{L}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathsf{L}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathsf{L}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and the other three infinitesimal generators remains the same. Where we use the notation $\varepsilon_{ijk} = \varepsilon^0{}_{ijk}$. We choose to work with the so-called fundamental basis

$$\left| \begin{array}{rcl} \mathsf{T}_i^{IJ} & = & -\varepsilon^{0iIJ} \\ \mathsf{T}_{(i+3)}^{IJ} & = & 2\eta^{[iI}\eta^{0]J} = \eta^{iI}\eta^{0J} - \eta^{0I}\eta^{iJ} \end{array} \right. \tag{3.25}$$

For example, the $\mathfrak{so}(4)$-valued 2-form $\mathbf{B}$ is decomposed as: $\mathbf{B} = \mathfrak{P}^{\triangleleft}\mathbf{B} + \mathfrak{P}^{\triangleright}\mathbf{B}$ so that the components decomposition is written: $\mathbf{B}^{IJ} = \big(\mathfrak{P}^{\triangleleft}\mathbf{B}\big)^{IJ} + \big(\mathfrak{P}^{\triangleright}\mathbf{B}\big)^{IJ}$, which we equivalently write $\mathbf{B}^{IJ} = \big(\mathfrak{P}^{\triangleleft}\mathbf{B}\big)_i^{IJ}\big(\mathbf{B}^{\triangleleft}\big)^i + \big(\mathfrak{P}^{\triangleright}\mathbf{B}\big)_i^{IJ}\big(\mathbf{B}^{\triangleright}\big)^i$. Generally, the projectors $\mathfrak{P}^{\triangleleft}$ and $\mathfrak{P}^{\triangleright}$ are defined:

$$\left| \begin{array}{rcl} \big(\mathfrak{P}^{\triangleleft}\big)^{IJ}{}_{KL} & = & (1/2)\big(\mathbb{I} - (1/2)\varepsilon^{IJ}{}_{KL}\big) = (1/2)\big(\delta^{IJ}_{KL} - (1/2)\varepsilon^{IJ}{}_{KL}\big) \\ \big(\mathfrak{P}^{\triangleright}\big)^{IJ}{}_{KL} & = & (1/2)\big(\mathbb{I} + (1/2)\varepsilon^{IJ}{}_{KL}\big) = (1/2)\big(\delta^{IJ}_{KL} + (1/2)\varepsilon^{IJ}{}_{KL}\big) \end{array} \right. \tag{3.26}$$

The self-dual and anti-self-dual projectors (3.26) have the following properties:

$$\left| \begin{array}{rcl} \big(\mathfrak{P}^{\triangleleft}\big)^{IJ}{}_{KL} & = & (1/2)\big(\mathbb{I} - (1/2)\varepsilon^{IJ}{}_{KL}\big) = (1/2)\big(\delta^{IJ}_{KL} - (1/2)\varepsilon^{IJ}{}_{KL}\big) \\ \big(\mathfrak{P}^{\triangleright}\big)^{IJ}{}_{KL} & = & (1/2)\big(\mathbb{I} + (1/2)\varepsilon^{IJ}{}_{KL}\big) = (1/2)\big(\delta^{IJ}_{KL} + (1/2)\varepsilon^{IJ}{}_{KL}\big) \end{array} \right. \tag{3.27}$$

We have also the followings properties [16] [17] [24],

$$\left| \begin{array}{rcl} \delta^{ij}\big(\mathfrak{P}^{\triangleleft}\big)_i^{IJ}\big(\mathfrak{P}^{\triangleleft}\big)_j^{KL} & = & \boldsymbol{\sigma}\big(\mathfrak{P}^{\triangleleft}\big)^{IJKL} \\ \delta_{IJKL}\big(\mathfrak{P}^{\triangleleft}\big)_i^{IJ}\big(\mathfrak{P}^{\triangleleft}\big)_j^{KL} & = & \delta_{ij} \\ \frac{1}{2}\varepsilon_{IJKL}\big(\mathfrak{P}^{\triangleleft}\big)_i^{IJ}\big(\mathfrak{P}^{\triangleleft}\big)_j^{KL} & = & \boldsymbol{\sigma}\big(\mathfrak{P}^{\triangleleft}\big)^{IJKL} \end{array} \right. \tag{3.28}$$

Now, if we introduce, the $\mathfrak{su}(2)$-Lie algebra indices $i = 1, 2, 3$, we obtain the following new tensor

$$\left| \begin{array}{rcl} \big(\mathfrak{P}^{\triangleleft}\big)_i^{IJ} & = & 2\big(\mathfrak{P}^{\triangleleft}\big)_{0i}^{IJ} \\ \big(\mathfrak{P}^{\triangleright}\big)_i^{IJ} & = & 2\big(\mathfrak{P}^{\triangleright}\big)_{0i}^{IJ} \end{array} \right. \quad \text{so that} \quad \left| \begin{array}{rcl} \big(\mathfrak{P}^{\triangleleft}\big)_i^{IJ} & = & \big(\delta^{IJ}_{0i} - (1/2)\varepsilon^{IJ}{}_{0i}\big) \\ \big(\mathfrak{P}^{\triangleright}\big)_i^{IJ} & = & \big(\delta^{IJ}_{0i} + (1/2)\varepsilon^{IJ}{}_{0i}\big) \end{array} \right. \tag{3.29}$$

We have the following properties:

$$\star\big(\mathfrak{P}^{\triangleleft}\big)_i^{IJ} \;=\; \big(\mathfrak{P}^{\triangleleft}\big)_i^{IJ} \qquad \star\big(\mathfrak{P}^{\triangleright}\big)_i^{IJ} \;=\; -\big(\mathfrak{P}^{\triangleright}\big)_i^{IJ} \tag{3.30}$$

in the case where we consider the Lie algebra $\mathfrak{so}(4)$. Notice that if we work with $\mathfrak{so}(1,3)$, we obtain:

$$\star\big(\mathfrak{P}^{\triangleleft}\big)_j^{IJ} \;=\; i\big(\mathfrak{P}^{\triangleleft}\big)_j^{IJ} \qquad \star\big(\mathfrak{P}^{\triangleright}\big)_j^{IJ} \;=\; -i\big(\mathfrak{P}^{\triangleright}\big)_j^{IJ} \tag{3.31}$$

3.3 Simplicity constraints

Simplicity constraints are written:

$$\varepsilon^{\mu\nu\rho\sigma}\mathbf{B}^{IJ}_{\mu\nu}\mathbf{B}^{KL}_{\rho\sigma} = \boldsymbol{\varkappa}\varepsilon^{IJKL} \qquad \text{with} \qquad \boldsymbol{\varkappa} = \frac{1}{4!}\epsilon^{\mu\nu\rho\sigma}\varepsilon_{IJKL}\mathbf{B}^{IJ}_{\mu\nu}\mathbf{B}^{KL}_{\rho\sigma} \tag{3.32}$$

Notice that we equivalently write: $\boldsymbol{\varkappa} = \frac{1}{4!}2\langle \mathbf{B} \wedge \star\mathbf{B}\rangle$ We write;

$$\begin{aligned} \mathbf{B}\wedge\star\mathbf{B} &= \left[\mathbf{B}^{IJ}\otimes\mathbf{\Delta}_{IJ}\right]\wedge\left[\star\left[\mathbf{B}^{KL}\otimes\mathbf{\Delta}_{KL}\right]\right] \\ &= \mathbf{B}^{IJ}\wedge\star\mathbf{B}^{KL}\otimes[\mathbf{\Delta}_{IJ},\mathbf{\Delta}_{KL}] \\ &= \left[\frac{1}{2}\mathbf{B}^{IJ}_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\otimes\mathbf{\Delta}_{IJ}\right]\wedge\star\left[\frac{1}{2}\mathbf{B}^{KL}_{\rho\sigma}\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma\otimes\mathbf{\Delta}_{KL}\right] \end{aligned} \tag{3.33}$$

Also, the object:

$$\langle\mathbf{B}\wedge\star\mathbf{B}\rangle = \langle\mathbf{B}^{IJ}_{\mu\nu}\mathbf{\Delta}_{IJ}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\star\left[\mathbf{B}^{IJ}_{\mu\nu}\mathbf{\Delta}_{IJ}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\right]\rangle \tag{3.34}$$

We equivalently notice that the simplicity constraints are written: (from now we consider the case where $\mathfrak{g} = \mathfrak{so}(4)$, so that $(-1)^{\boldsymbol{\sigma}} = 1$

$$\mathbf{B}^{IJ}\wedge\mathbf{B}^{KL} = \left[\frac{1}{4!}\epsilon^{\mu\nu\rho\sigma}\varepsilon_{MNOP}\mathbf{B}^{MN}_{\mu\nu}\mathbf{B}^{OP}_{\rho\sigma}\right]\varepsilon^{IJKL} \tag{3.35}$$

There are four sectors of solutions: two of them are of topological nature ($\mathbf{B}^{IJ} = \pm\mathfrak{U}^{IJ}$), the others are of gravitational nature ($\mathbf{B}^{IJ} = \pm\star(\mathfrak{U}^{IJ})$).

$$\left|\begin{aligned} \mathbf{B}^{IJ} &= e^I\wedge e^J \\ \mathbf{B}^{IJ} &= -e^I\wedge e^J \end{aligned}\right. \qquad\qquad \left|\begin{aligned} \mathbf{B}^{IJ} &= \star(e^I\wedge e^J) \\ \mathbf{B}^{IJ} &= -\star(e^I\wedge e^J) \end{aligned}\right. \tag{3.36}$$

3.4 Urbantke metric

3.4.1 Urbantke metric from $\mathfrak{su}(2)$-valued 2-form B

The Urbantke formula [52] [53] allows us to construct a metric $g^{\mathsf{Urb}}_{\mu\nu}$ from a $\mathfrak{su}(2)$-valued 2-form $\mathbf{B}\in\Omega^2(\mathcal{X})\otimes\mathfrak{su}(2)$. Urbantke demonstrate the formula:

$$\sqrt{g^{\mathsf{Urb}}}g^{\mathsf{Urb}}_{\mu\nu} = \frac{1}{12}\varepsilon_{ijk}\varepsilon^{\alpha\beta\gamma\delta}\mathbf{B}^i_{\mu\alpha}\mathbf{B}^j_{\beta\gamma}\mathbf{B}^k_{\delta\nu} \tag{3.37}$$

where ε is the unique singlet in the tensor product of three adjoint representations of $\mathfrak{su}(2)$.

3.4.2 Urbantke metric from $\mathfrak{so}(4)$ or $\mathfrak{so}(1,3)$-valued 2-form B

In the context of the non-chrial formulation, we are interested by the Lie algebra $\mathfrak{so}(4)$ or the Lie algebra $\mathfrak{so}(1,3)$. This corresponds to a generalization of the Urbantke formula where we need to consider the tensorial product of three adjoint representations of the Lie algebra $\mathfrak{so}(4)$. (We leave apart the Lorentzian case for the moment). The tensor product of three adjoint representations is given with 2 singlets, a basis of this two dimensional space is given by: $\delta_{N[I}\delta_{J]MKL}$ and $\delta_{N[I}\varepsilon_{J]MKL}$ where $\delta_{IJKL} = \frac{1}{2}\big(\delta_{IK}\delta_{JL} - \delta_{IL}\delta_{JK}\big)$. In this case, where the Lie algebra under consideration is $\mathfrak{so}(4)$, we have two Urbantke metrics, a left handed and right handed one. $\big(g^{\mathsf{Urb}}_{\mu\nu}\big)^{\triangleleft}$ and $\big(g^{\mathsf{Urb}}_{\mu\nu}\big)^{\triangleright}$ such that:

$$\sqrt{\big(g^{\mathsf{Urb}}_{\mu\nu}\big)^{\triangleleft}}\big(g^{\mathsf{Urb}}_{\mu\nu}\big)^{\triangleleft} = \frac{1}{12}\Big(\delta_{IN}\big(\delta_{JMKL} - \frac{1}{2}\varepsilon_{JMKL}\big)\Big)\varepsilon^{\alpha\beta\gamma\delta}\mathbf{B}^{IJ}_{\mu\alpha}\mathbf{B}^{KL}_{\beta\gamma}\mathbf{B}^{MN}_{\delta\nu} \tag{3.38}$$

and

$$\sqrt{\big(g^{\mathsf{Urb}}_{\mu\nu}\big)^{\triangleright}}\big(g^{\mathsf{Urb}}_{\mu\nu}\big)^{\triangleright} = \frac{1}{12}\Big(\delta_{IN}\big(\delta_{JMKL} + \frac{1}{2}\varepsilon_{JMKL}\big)\Big)\varepsilon^{\alpha\beta\gamma\delta}\mathbf{B}^{IJ}_{\mu\alpha}\mathbf{B}^{KL}_{\beta\gamma}\mathbf{B}^{MN}_{\delta\nu} \tag{3.39}$$

3.5 BF theory, Holst term and further developments

Notice that further consideration on the BF formalism presented in the previous sections has been investigate by Smolin [44] (Unification of gravity and Yang-Mills theory), Smolin and Speziale [45] (role of the Immirzi parameter), the work of Alexandrov, Krasnov (Hamiltonian analysis) [2].

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