A path towards Order and Disorder

Non–linear dynamics, chaos and self-organization in some natural theories

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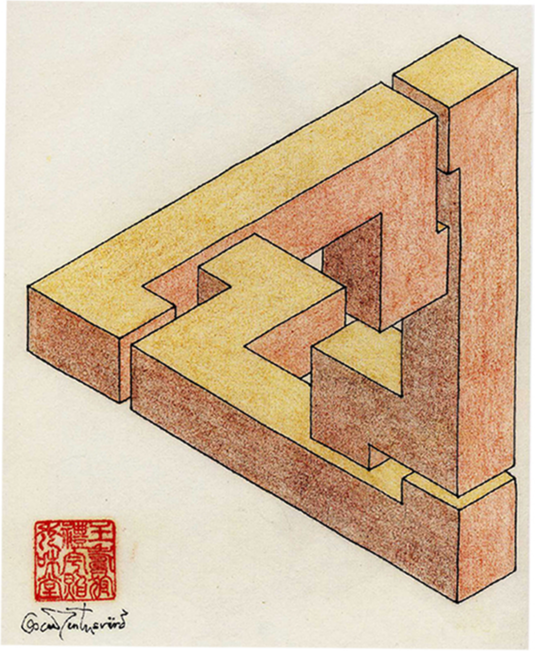
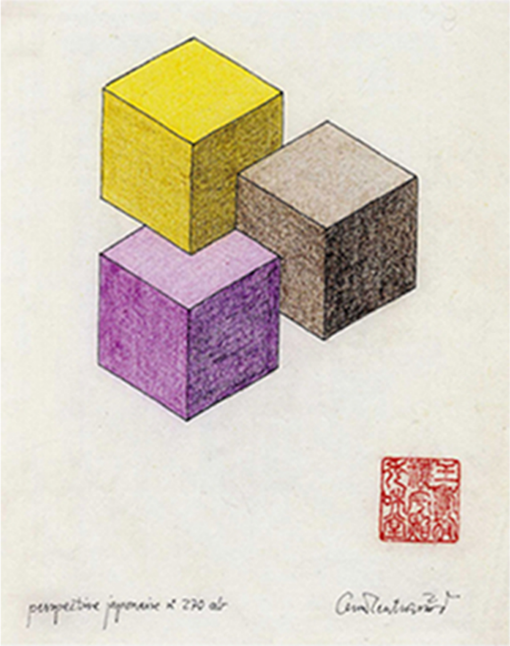
$\boxed{2}$ Self–similarity

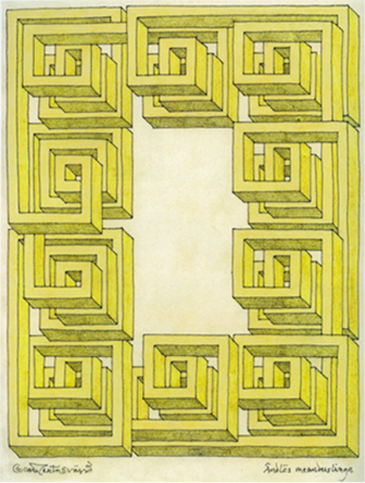
- Impossible geometry.
- Self–similarity, Tessa,
- Fractals – Fractal dimension
- Julia set – Mandelbrot –

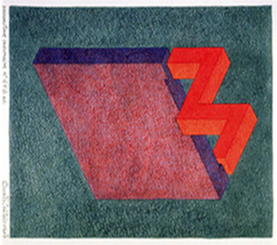
O. Reutersvärd

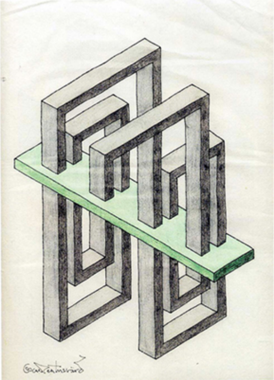
- Oscar Reutersvrd Swedish artist (1915 – 2002)
- created hundreds of different impossible figures
- first impossible triangle in (1934)




perspective japonaise n° 270 ab


Ändlös meanderslinga

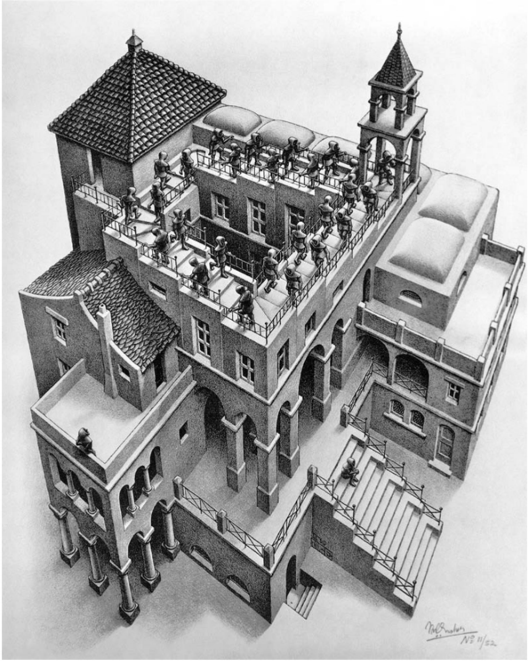


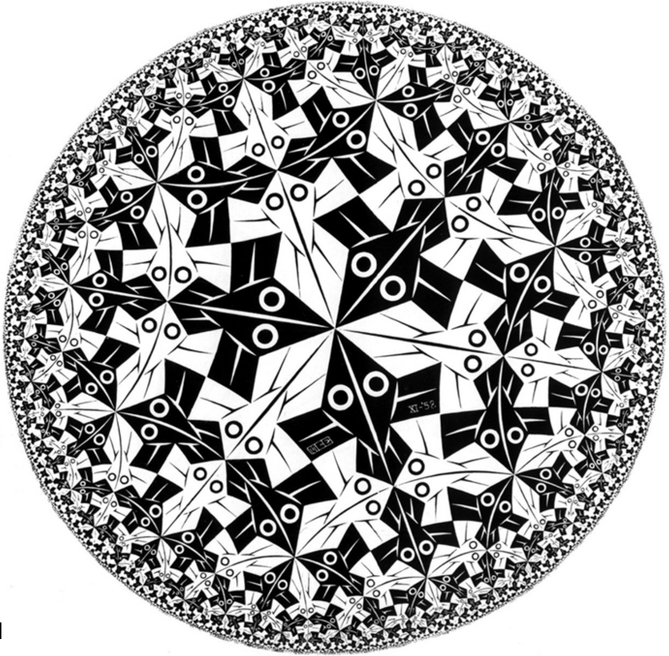


M.C. Escher

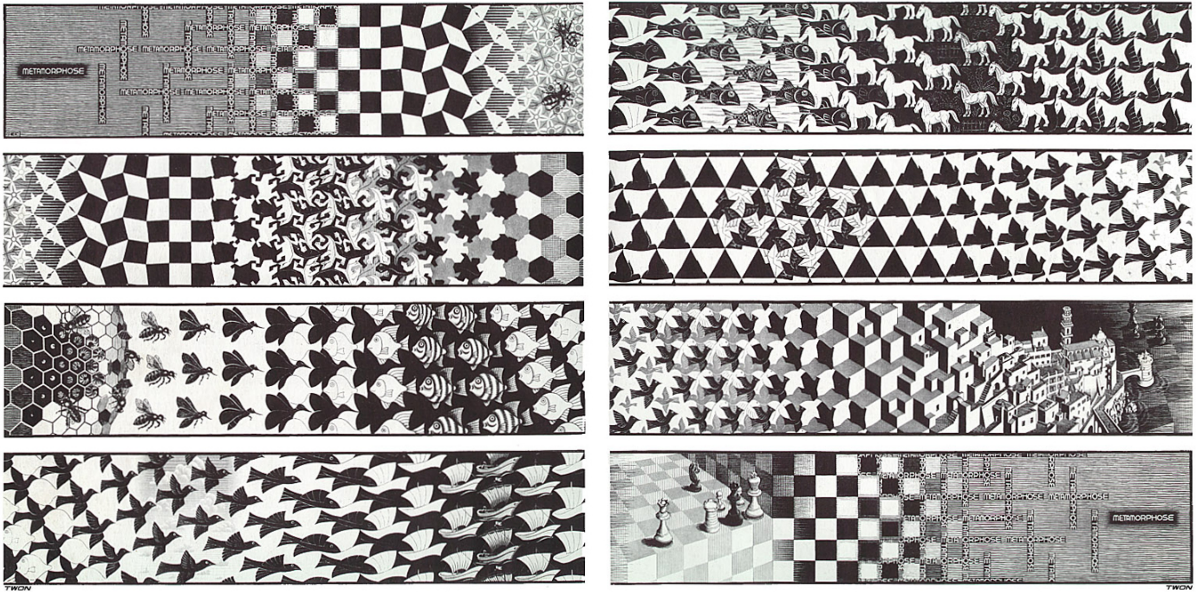
- Maurits Cornelis Escher (1898 – 1972), Dutch graphic artist
- Mathematically-inspired work — tessellations

Ascending and Descending, M. C. Escher. Lithograph, 1960 →

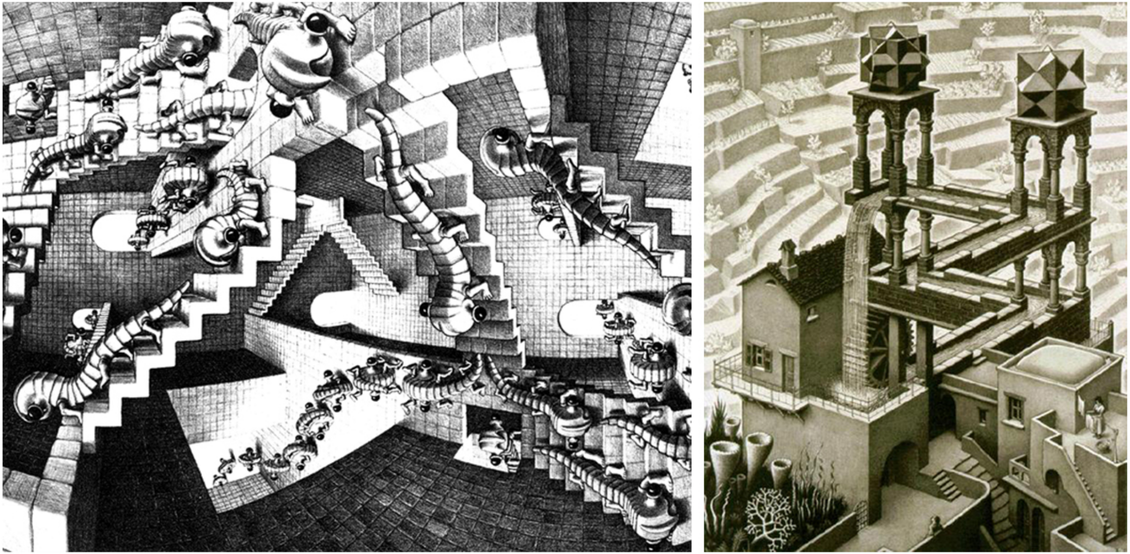




Esher — Circle limit III



Esher — metamorphosis III — (1968)

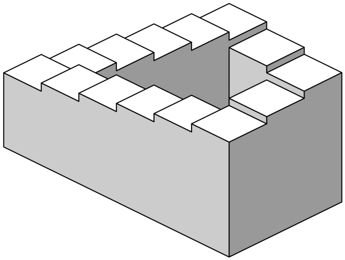


Esher — Relativity (left), Waterfall (right)

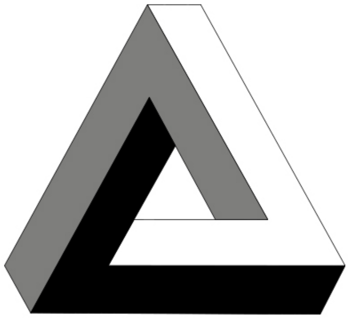
Impossible geometry: Roger Penrose

- Sir Roger Penrose (1931 –), English mathematical physicist, mathematician and philosopher of science
- Penrose triangle (popularised it in the 1950).
- Penrose staircase (1958).

Penrose impossible staircase



Penrose triangle

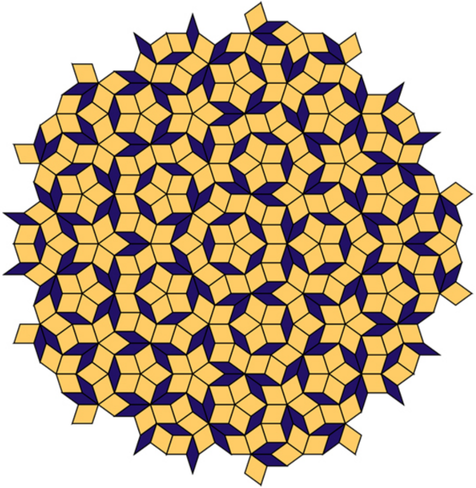


Penrose, L.S.; Penrose, R. (1958). "Impossible objects: A special type of visual illusion". British Journal of Psychology 49 (1): 31?33.

Penrose tessellations, Penrose tiling

Penrose tiling has many remarkable properties:

- Non-periodic, (lacks any translational symmetry).
- Self-similar, so the same patterns occur at larger and larger scales. T
- It is a quasicrystal: implemented as a physical structure a Penrose tiling will produce Bragg diffraction (fivefold symmetry).



Paradoxical loop of causality

"impossible staircase willed itself into existence in a paradoxical loop of causality (J.C. Baez)

Escher inspired Penrose

At an Escher conference in Rome in 1985, Roger Penrose said that he had been greatly inspired by Escher's work when he and his father discovered the Penrose stairs. Penrose said:

- had first seen Escher's work at a conference in Amsterdam in 1954
- was "absolutely spellbound",
- finally arrived at the impossible Penrose triangle.

Penrose inspired Escher

Escher (letter to his son January 1960)

"... working on the design of a new picture, which featured a flight of stairs which only ever ascended or descended, depending on how you saw it. They form a closed, circular construction, rather like a snake biting its own tail. And yet they can be drawn in correct perspective: each step higher (or lower) than the previous one. I discovered the principle in an article which was sent to me, and in which I myself was named as the maker of various 'impossible objects'. But I was not familiar with the continuous steps of which the author had included a clear, if perfunctory, sketch..."

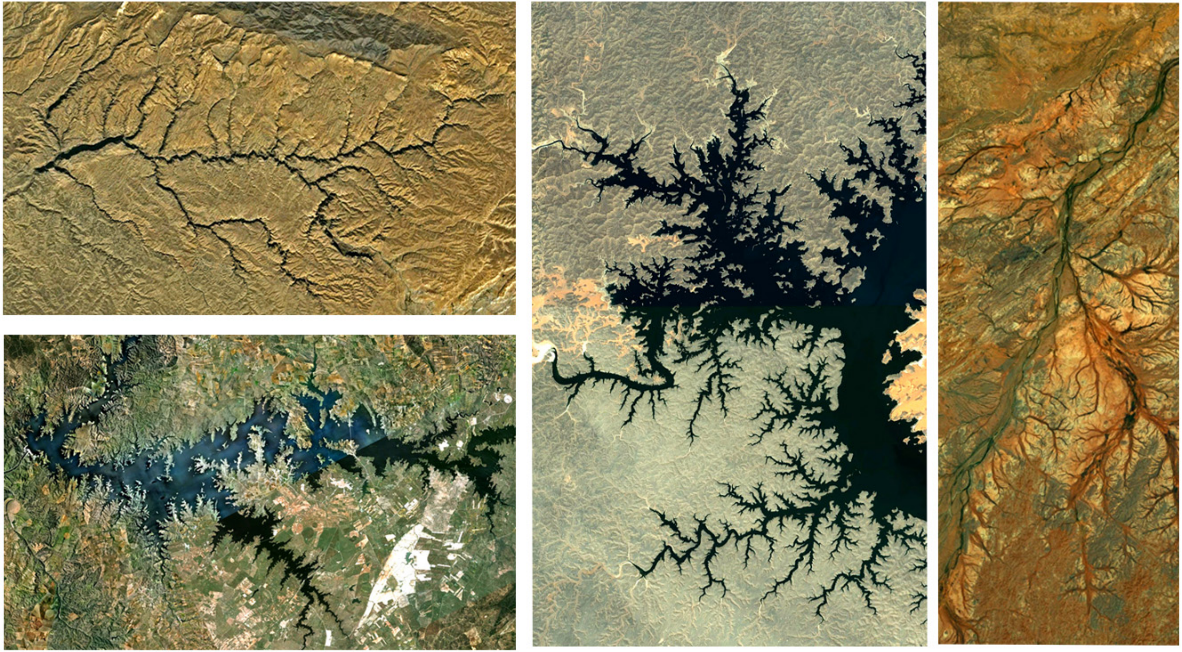
Fractal curves, fractal objects and fractal dimension

Fractal objects:

- **Self-similar** (or quasi self-similar)
- **Fine structure** on arbitrary small scales.
- ofter have simple, **recursive** definitions

- **1874 — G. Cantor** Cantor set, prototype of a fractal
- **1904 — H. von Koch**, von Koch curve,
- **1915 — W. Sierpiński**: constructed the Sierpiński Triangle and Sierpiński carpet (1916).
- **1918 — F. Hausdorff**, Hausdorff dimension.
- **1918 — P. Fatou and J. Gaston** French mathematicians,
- **1975 — B. Mandelbrot.** coined the terms *fractal dimension* and *fractal*
- **1980 — B. Mandelbrot.**
Benoit Mandelbrot, Fractal aspects of the iteration of $z \mapsto \lambda z(1-z)$ for complex λ, z Annals NY Acad. Sci. 357, 249/259

Sark island — Channel Islands in the southwestern English Channel




Fractal Dimension

- Based on Benoit Mandelbrot's (1967) paper on self-similarity in which he discussed fractional dimensions
- Benoit B. Mandelbrot (1983). The fractal geometry of nature. Macmillan. ISBN 978-0-7167-1186-5. Retrieved 1 February 2012

Mathematical definition

The fractal dimension D is given by introducing the following two numbers:

- $\mathcal{N}$ stands for the number of new sticks
- $\mathcal{S}$ stands for the scaling factor

The fractal dimension is given by:

$$D := \frac{\log \mathcal{N}}{\log \mathcal{S}}$$

Cantor Set: prototype of a fractal

- discovered by H.J.S. Smith (1874)
- introduced by Georg Cantor (1883)

The Cantor set has the following properties:

- a perfect set, nowhere dense, uncountable
- Self-similarity
- complete metric space and (via Heine–Borel theorem) compact.
- natural Haar measure, zero Lebesgue measure.

Fractal dimension

The Cantor set has fractal dimension D_{Cantor}:

$$D_{\text{Cantor}} \simeq 0.631$$

Smith-Volterra-Cantor Set

Built by removing a central interval of length 2^{-2n} of each remaining interval at the n–th iteration. It has the following properties:

- a perfect set, nowhere dense, uncountable
- Self-similarity
- **Non–zero** Lebesgue measure. $(1/2)$

Hausdorff dimension

The Smith-Volterra-Cantor set has Hausdorff dimension $D_{\text{Smith-Volterra-Cantor}}$:

$$D_{\text{Smith-Volterra-Cantor}} = 1.$$

Generalized Cantor Set

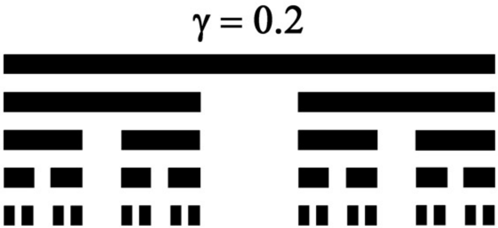
Fractal dimension

Generalized Cantor set has fractal dimension:

$$D_{\text{Cantor}} = \frac{-\log(2)}{\log\left(\frac{1-\gamma}{2}\right)}$$

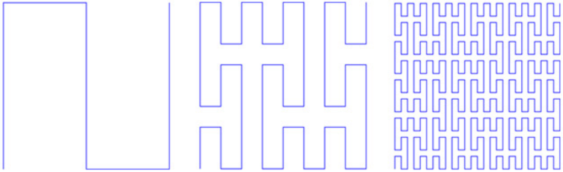
Built by removing at the m–th iteration the central interval of length $\gamma\, l_{m-1}$ from each remaining segment (of length $l_{m-1} = (1-\gamma)^{m-1}/2^{m-1}$).

- At $\gamma = 1/3$ one obtains the usual Cantor set.
- Varying γ between 0 and 1 yields any fractal dimension $0 < D < 1$.



Paeno and Hilbert curves

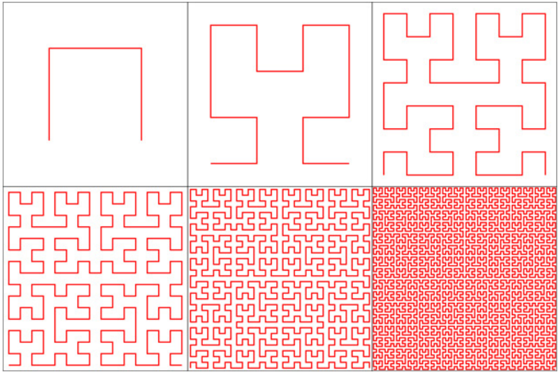
Paeno curve



- a space-filling curve discovered by Giuseppe Peano (1890).
- The Paeno curve has topological and fractal dimensions, $D^{\text{To}}_{\text{Paeno}}$ and D_{Paeno}:

$$D^{\text{To}}_{\text{Paeno}} = 1, \qquad D_{\text{Paeno}} = 2.$$

Hilbert curve



The Hilbert curve has topological and Hausdorff dimensions, $D^{\text{To}}_{\text{Hilbert}}$ and D_{Hilbert}:

$$D^{\text{To}}_{\text{Hilbert}} = 1, \qquad D_{\text{Hilbert}} = 2.$$

von Kock snowflake

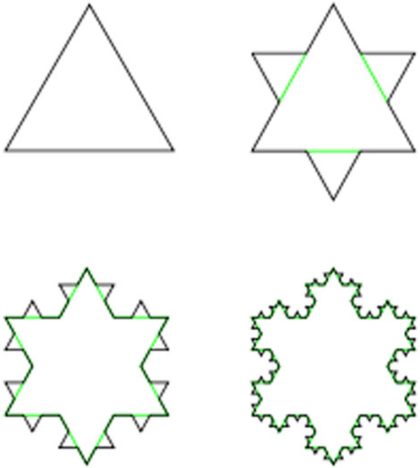
- Swedish mathematician Helge von Koch
- 1904 paper titled "Sur une courbe continue sans tangente, obtenue par une construction géométrique lmentaire"

Curve of infinite perimeter enclosing a finite area.

- **Perimeter**: $\mathcal{P}_n := 3 \cdot \ell \cdot (\frac{4}{3})^n$, ℓ is the size of the original triangle.
- **Area**: $\mathcal{A}_n := \frac{\mathcal{A}_0}{5}\left(8 - 3(\frac{4}{9})^n\right)$ so that $\mathcal{A}_\infty :=$
- Limite of perimeter $\mathcal{P}_\infty := \infty$
- Limite of area $\lim_{n\to\infty} \mathcal{A}_n = \frac{8}{5} \cdot \mathcal{A}_0$

Fractal dimension of von Kock curve

$$D_{\text{von Kock curve}} = \frac{\log(4)}{\log(3)} \simeq 1.26186$$

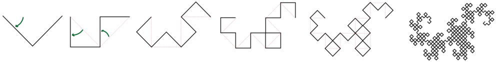


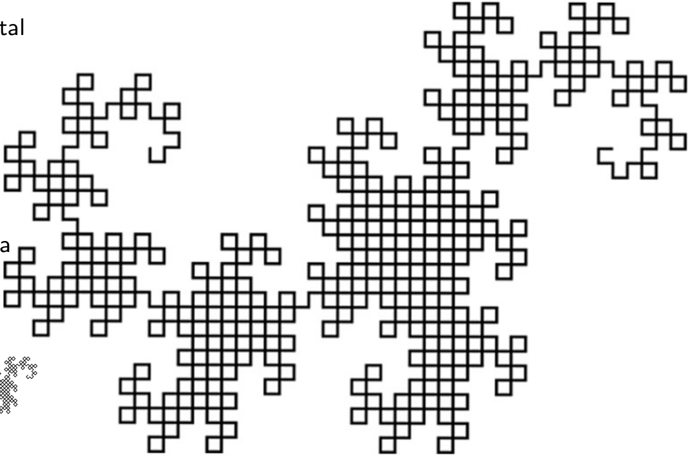
Dragon-like curves

- any member of a family of self-similar fractal curves, which can be approximated by recursive methods
- Notion of Lindenmayer systems.

Construction:

- Starting from a base segment,
- replace each segment by 2 segments with a right angle and with a rotation of 45 alternatively to the right and to the left:





Dragon-like curves

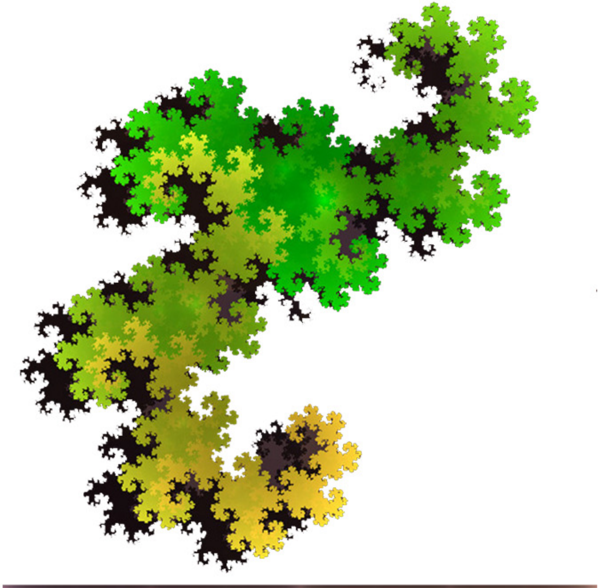
Fractal dimension of the Dragon curve

$$D_{\text{von Kock curve}} = \frac{\log(2)}{\log(\sqrt{2})} = 2$$

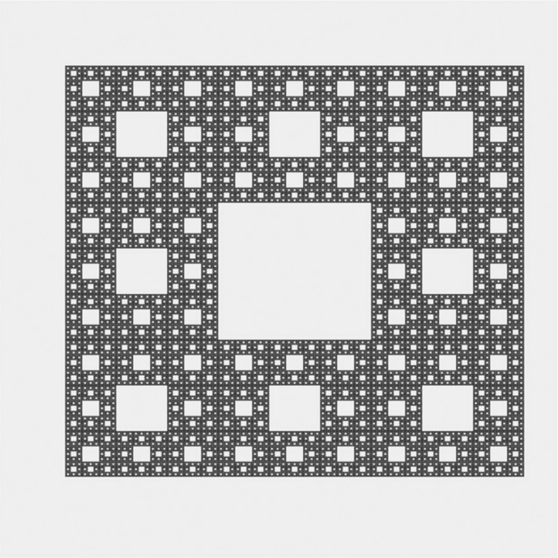
Boundary of the Dragon curve

- has an infinite length,
- The fractal dimension has been approximated numerically by Chang and Zhang

$$\overline{D}_{\text{Dragon}} \cong 1.523627086202492.$$

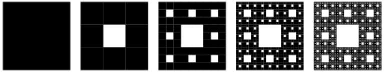


Sierpinski carpet



Curve of infinite perimeter enclosing a finite area.

- Limite of perimeter $\mathcal{P}_\infty := \infty$
- Limite of area $\lim_{n\to\infty} \mathcal{A}_n = 0$



Fractal dimension

$$D_{\text{Sierpinski carpet}} = \frac{\log(8)}{\log(3)} \cong 1.89$$

Mandelbrot set: mathematical definition

The Mandelbrot $\mathfrak{M}$ set is defined by a family of complex quadratic polynomials

$$\begin{array}{rcl} P_c : \mathbb{C} & \to & \mathbb{C} \\ z & \mapsto & P_c(z) := z^2 + c \end{array}$$

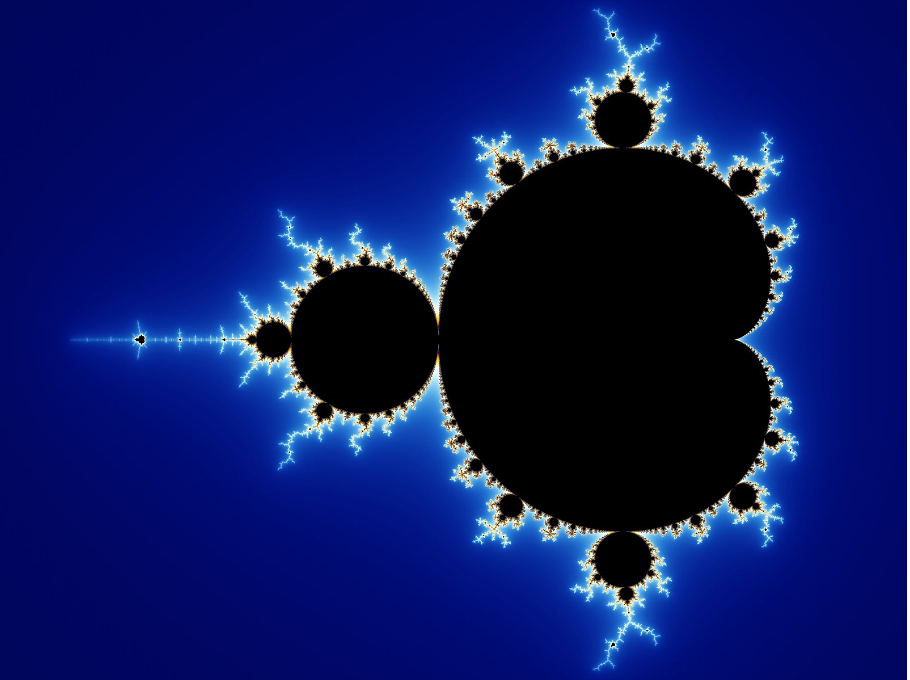
For any $c \in \mathbb{C}$, consider the sequence

$$(0, P_c(0), P_c(P_c(0)), P_c(P_c(0)), \dots)$$

The Mandelbrot set $\mathfrak{M} \subset \mathbb{C}$ is

$$\mathfrak{M} := \{c \in \mathbb{C}; \forall n \in \mathbb{N}, \left|P_c^n(0)\right| < \infty\}$$

In fact $\mathfrak{M} := \{c \in \mathbb{C}; \forall n \in \mathbb{N}, \left|P_c^n(0)\right| < 2\}$



$\longleftarrow$ Drawing of the Mandelbrot set, with a continuous choice of colors.

Fractal dimension

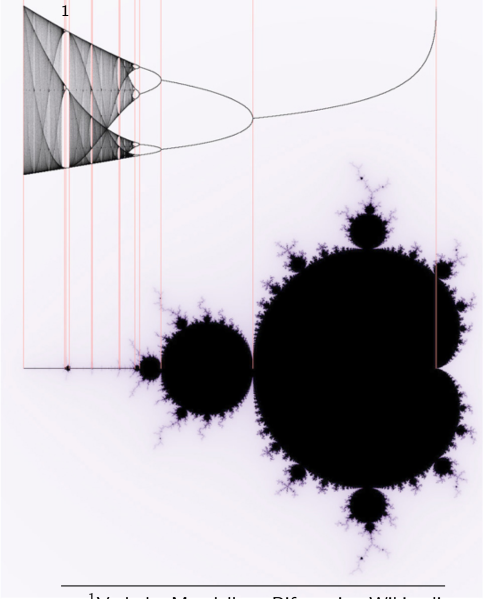
The Mandelbrot set has fractal dimensions:

$$D_{\text{Mandelbrot}} = 2$$

Its boundary has fractal dimension (M. Shishikura)

$$\overline{D}_{\text{Mandelbrot}} = 2.$$

Properties of the Mandelbrot set $\mathfrak{M}$



- $\mathfrak{M}$ is connected (J. Hubbard and A. Douady)
- $\mathfrak{M}$ contains many *baby Mandelbrot sets*
- $\mathfrak{M} \cap \mathbb{R} \subset [-2, 0.25]$.
- Parameters along $\mathfrak{M} \cap \mathbb{R}$ in one-to-one correspondence with those of the real logistic family,

$$x \mapsto \lambda x\,(1-x)\,, \quad \lambda \in [1,4].$$

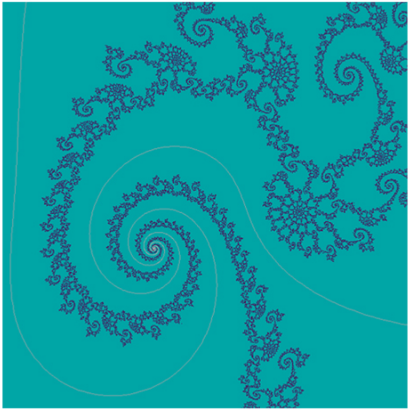
where $c = \frac{\lambda}{2}\left(1 - \frac{\lambda}{2}\right)$

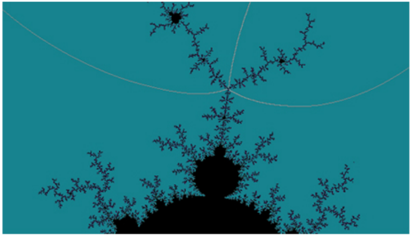
- $\overline{\mathfrak{M}}$ is exactly the bifurcation locus of the quadratic family.

A. Douady and J.H. Hubbard , *On the dynamics of polynomial-like mappings*, Ann. Sci. Ecole Norm. Sup. (4) 18 (1985), no. 2, 287?343. MR816367 (87f:58083)

Peter Haïssinsky, Modulation dans l'ensemble de Mandelbrot, The Mandelbrot set, theme and variations, 2000, pp. 37–65. MR1765084 (2002c:37067)

[1] Verhulst-Mandelbrot-Bifurcation Wikipedia,





The Mandelbrot set is self-similar under magnification in the neighborhoods of the Misiurewicz points.

Non-Branch points

- Point $c = M_{23,2}$
- at $c = -0.77568377 + 0.13646737 \cdot i$, is near a Misiurewicz point $M_{23,2}$.
- **a center of a two-arms spiral**

Branch points

- $c = M_{4,1}$, Point $c = -0.1010... + 0.9562... \cdot i = M_{4,1}$, is a principal Misiurewicz point of the $1/3$ limb. It has 3 external rays $9/56, 11/56 and 15/56$.

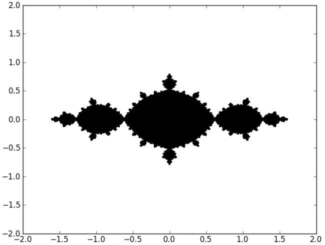
Julia sets mathematical definition

The Julia J sets is defined by a family of complex quadratic polynomials

$$\begin{aligned} P_{\mathrm{c}} : \mathbb{C} &\rightarrow \mathbb{C} \\ \mathrm{z} &\mapsto P_{\mathrm{c}}(\mathrm{z}) := \mathrm{z}^2 + \mathrm{c} \end{aligned}$$

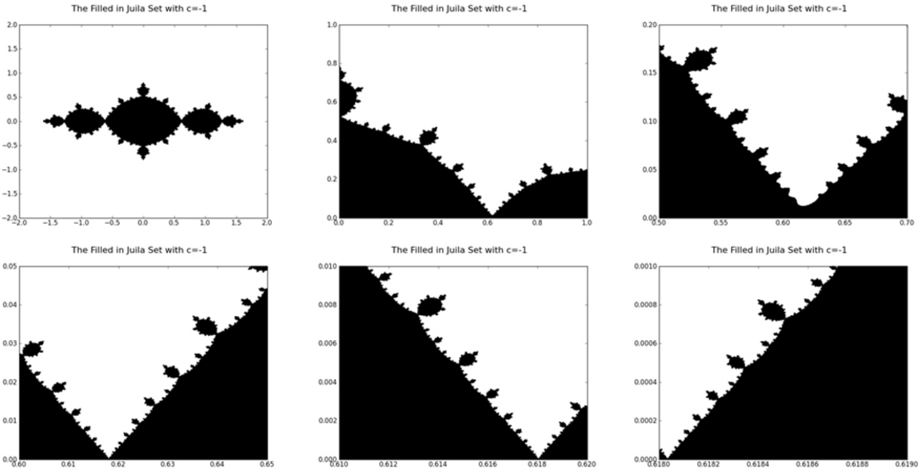
Julia set

- for almost any z, this iteration generate a fractal (e.g. not true for $\mathrm{z} = 0, -2$
- the Mandelbrot set is defined as the set of all c such that the **Julia set is connected**.
- For parameters outside the Mandelbrot set, the Julia set is a Cantor space

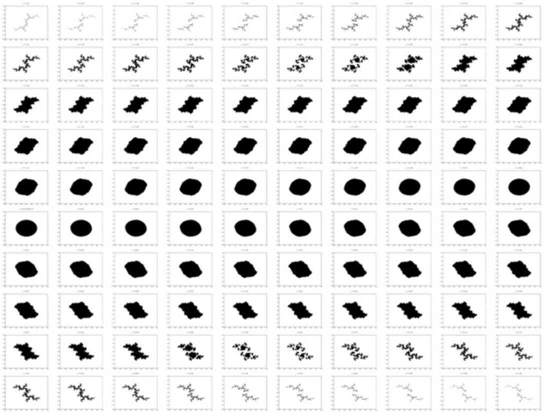


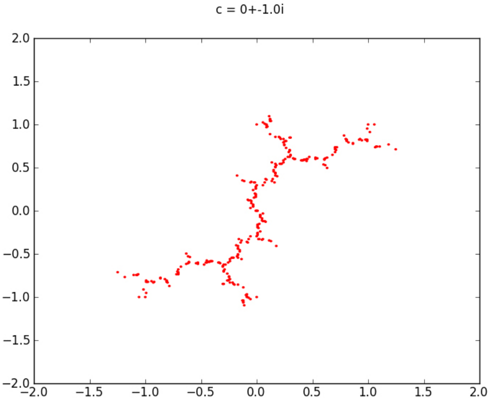
Julia set with $c = -1$.

Zoom in the Julia set



← Zooing into the Julia set related to $c = -1$



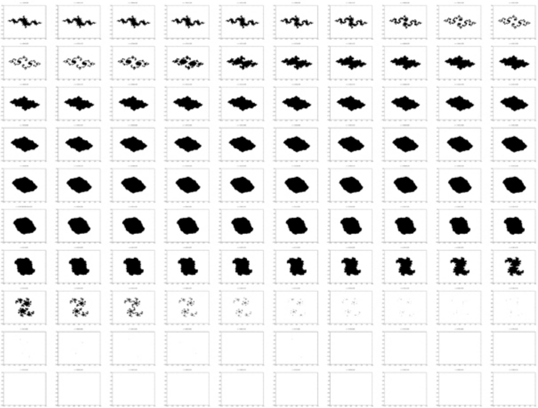


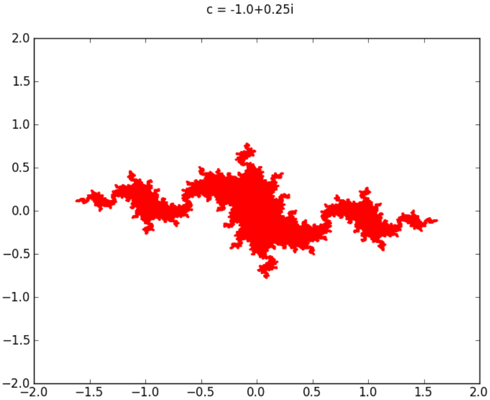
Sequence of Julia sets

- Start from $c = -i$
- We draw the difference Julia sets with

$$-i < c < i$$

with steps of $\ell := 0.02i$.



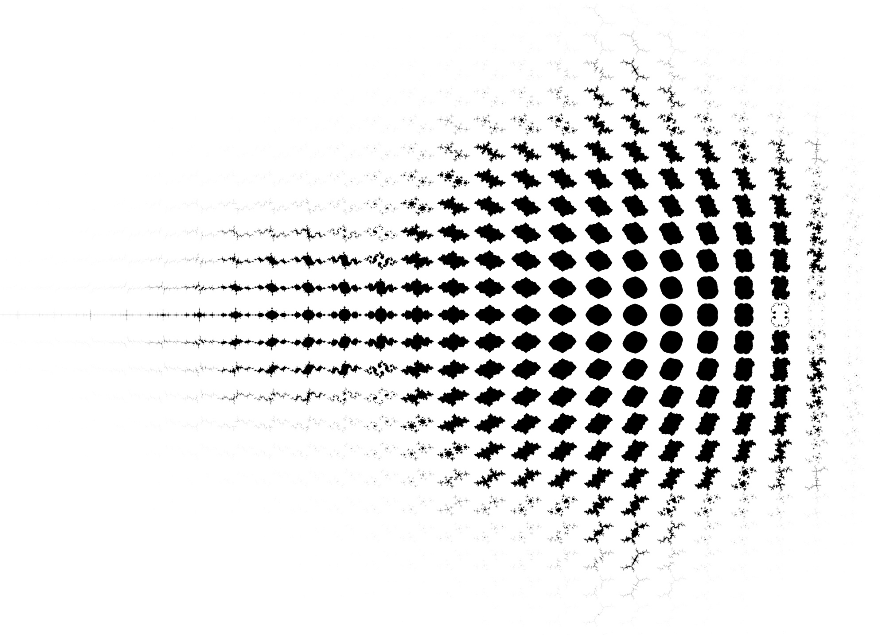


Sequence of Julia sets

- Start from $c = 1 + 0.25i$
- We draw the difference Julia sets with

$$1 + 0.25i < c < -1 + 0.25i$$

with steps of $\ell := 0.02i$.



$\leftarrow$ Drawing of the Mandelbrot set, with a continuous choice of colors.

Fractal dimension

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$$D_{\text{Mandelbrot}} = 2$$

Its boundary has fractal dimension

$$\overline{D}_{\text{Mandelbrot}} = 2.$$