Generalized Hamiltonian Gravity

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General Relativity: 100 years after Hilbert Stará Lesná, Slovakia. [21.08.2015]

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Fundamental feature of General Relativity

The shape of space-time and the structures which determine its physical properties (i.e. the metric, the connection) are not fixed a priori, but by the dynamics

However if we lift the description of this theory to the principal bundle of orthonormal moving frames over the space-time, we need to assume **a priori constraints** on its total space:

- axioms of the definition of a principal bundle
- axioms of the definition of a connection

WEC (Palatini) formulation of gravity

Analysis of the **Hamiltonian structure** of Einstein equations, starting from the variational Weyl–Einstein–Cartan (WEC) formulation (also termed Palatini)

Multisymplectic approach

- respects in a natural way the locality of physical theories
- as covariant as possible, we look for a formulation which depends in a minimal way on choices of coordinates

However, for a gauge theory ...

- choosing a particular gauge
- in the WEC formulation it corresponds to choose a particular moving frame.

In the standard description of gauge theories the coordinates independence of the multisymplectic formalism is spoiled by the need for choosing a particular gauge (here a particular moving frame) for writing the equations.

Outlooks - Table of contents

Then,

- Lift the problem on the total space of the principal bundle. (Ehresmann).
- ideas, points of view developed by Cartan, (the equivalence problem)

Multisymplectic formulation on the total space of the principal bundle of orthonormal frames on the 4-dimensional space-time.

The approach we follow releases the 10-dimensional total space of the principal bundle and the connection from these *a priori* constraints of a **principal fiber bundle** and of a **principal connection**.

1 Gauge Field

- Gauge Field: Fibration + Connection.
- Normalization and equivariance property

Gauge Field = Fibration + Connection

(Principal) Fibration

Fibration hypothesis: identification of the space-time entity as the base space of some fiber bundle **over** the space-time manifold.

- continuous surjection $\pi : \mathcal{P} \to \mathcal{X}$ between two topological spaces $\mathcal{X}$ and $\mathcal{P}$.
- $\mathcal{P}_x := \pi^{-1}(x)$ is the fiber **over** some point of the base manifold
- Local triviality conditions
- Right action $R_g : \mathcal{P} \times \mathfrak{G} \longrightarrow \mathcal{P}$

Connection

Connection hypothesis: introduction of a:

- connection ∇ on the space-time manifold
- its lift to the total space of the bundle $\boldsymbol{\nabla} := \pi^\star \nabla$.

A connection is encoded by a 1-form with value in the Lie algebra $\mathfrak{g}$:

- $A \in \Omega^1(\mathcal{X}) \otimes \mathfrak{g}$
- lifted to a connection on the total space of the bundle $\boldsymbol{\eta} := \pi^\star A \in \Omega^1(\mathcal{X}) \otimes \mathfrak{g}$

principal fiber bundle

Consider a principal fiber bundle $(\mathcal{P}, \mathcal{X}, \pi, \mathfrak{G})$, we have:

- **Right action**:

$$\begin{array}{llll} R_g : & \mathcal{P} \times \mathfrak{G} & \longrightarrow & \mathcal{P} \\ & (z, g) & \longmapsto & z \cdot g = R_g(z) \end{array}$$

- **Fundamental vector fields**: $\rho_\xi(z) = z \cdot \xi$.

$\forall z \in \mathcal{P}, \forall \xi \in \mathfrak{g}, \rho_\xi(z) := \mathrm{d}/\mathrm{d}t(z \cdot e^{t\xi})|_{t=0}$;

- **Vertical space** $V_z\mathcal{P} := \mathrm{kerd}(\pi_\mathcal{X})_z$.
- **Maurer-Cartan** canonical 1-form $\theta_z^{\mathrm{MC}} : V_z\mathcal{P} \longrightarrow \mathfrak{g}$ characterized by:

$$\begin{array}{ll} \forall z \in \mathcal{P}_x & [z \cdot \theta_z^{\mathrm{MC}}(v) = v, \quad \forall v \in V_z\mathcal{P}] \\ \Longleftrightarrow & [\theta_z^{\mathrm{MC}}(z \cdot \xi) = \xi, \quad \forall \xi \in \mathfrak{g}]. \end{array}$$

principal connection

$\Gamma(\mathcal{P}, \mathfrak{g} \otimes T^*\mathcal{P}) := \Omega^1(\mathcal{P}, \mathfrak{g})$ the space of sections of the bundle $\mathfrak{g} \otimes T^*\mathcal{P}$ over $\mathcal{P}$.

Normalization condition

$$\boldsymbol{\eta}_z|_{\mathrm{V}_z\mathcal{P}} = \theta_z^{\mathrm{MC}}, \quad \forall z \in \mathcal{P}.$$

Lie algebraic level: $\rho_\xi \,\lrcorner\, \boldsymbol{\eta} = \xi$

Equivariance condition

$$\forall (g, z) \in \mathfrak{G} \times \mathcal{P}, \ (R_g^*\boldsymbol{\eta})_z = \mathrm{Ad}_{g^{-1}} \circ \boldsymbol{\eta}_z = g^{-1}\boldsymbol{\eta}_z g,$$

Lie algebraic level $\mathcal{L}_{\rho_\xi}\boldsymbol{\eta} + [\xi, \boldsymbol{\eta}] = 0, \quad \forall \xi \in \mathfrak{g},$

Gauge principle, local gauge transformations

In a trivialization of $(\mathcal{P}, \mathcal{X}, \pi, \mathfrak{G})$

Consider $\boldsymbol{\sigma} : \mathcal{X} \longrightarrow \mathcal{P}$ be a section of $\mathcal{P}$.

$$\begin{array}{rccc} \tau : & \mathcal{P} & \longrightarrow & \mathcal{X} \times \mathfrak{G} \\ & z & \longmapsto & (x, g) \end{array}$$

where $x = \pi_{\mathcal{X}}(z)$ and $\boldsymbol{\sigma}(x) \cdot g = z$. Note that $\theta^{\mathsf{MC}}|_{\mathcal{P}_x} = g^{-1}\mathrm{d}g$.

Normalization condition

$$\begin{aligned} \boldsymbol{\eta}_{(x,g)} &= \boldsymbol{\eta}_i(x, g)(g^{-1}\mathrm{d}g)^i + \boldsymbol{\eta}_\mu(x, g)\mathrm{d}x^\mu \\ &= g^{-1}\mathrm{d}g + \boldsymbol{\eta}_\mu(x, g)\mathrm{d}x^\mu \end{aligned}$$

Equivariance condition

$$\boldsymbol{\eta}_{(x,g)} = g^{-1}\mathrm{d}g + g^{-1}A_x g, \text{ where } A_x := A_\mu(x)\mathrm{d}x^\mu.$$

Let $\boldsymbol{\sigma} : \mathcal{X} \longrightarrow \mathcal{P}$ and $\boldsymbol{\sigma}' : \mathcal{X} \longrightarrow \mathcal{P}$ be two sections.
there exists $\gamma : \mathcal{X} \longrightarrow \mathfrak{G}$ s.t. $\forall x \in \mathcal{X}$.

$$\boldsymbol{\sigma}'(x) = \boldsymbol{\sigma}(x) \cdot \gamma(x)$$

- The pull-back of $\boldsymbol{\eta}$ by $\boldsymbol{\sigma}$ is $\boldsymbol{\sigma}^*\boldsymbol{\eta} = A$
- The pull-back of $\boldsymbol{\eta}$ by $\boldsymbol{\sigma}'$ is $(\boldsymbol{\sigma}')^*\boldsymbol{\eta} = A'$

$$A' = \gamma^{-1}\mathrm{d}\gamma + \gamma^{-1}A\gamma$$

2 Gauge Gravity

- Weyl–Einstein–Cartan action, (Palatini formulation of vierbein gravity)
- Dynamical field: solder form, Lorentz connection, Cartan connection

Weyl–Einstein–Cartan action functional

Consider the **Weyl–Einstein–Cartan** (WEC) functional: (or **Palatini** functional)

$$\mathcal{S}_{\mathrm{WEC}}[e, \mathrm{A}] = \frac{\kappa}{2} \int \varepsilon_{abcd} e^a \wedge e^b \wedge F^{cd},$$

where $\kappa := (16\pi G)^{-1}$.

Dynamical fields: locally as being pairs (e, A),

- $e = (e^0, e^1, e^2, e^3)$ is a moving coframe on $\mathcal{X}$ (defining a metric $\mathrm{h}_{ab} e^a \otimes e^b$ on the tangent bundle $T\mathcal{X}$.)
- A is a $\mathfrak{g}$-valued connection 1-form on $\mathcal{X}$.

Einstein's equations are equivalent to the Euler-Lagrange system of equations

$$\begin{aligned} \mathbf{d}_{\mathrm{A}} e^a &= 0, \\ \varepsilon_{abcd} e^b \wedge F^{cd} &= 0, \end{aligned}$$

where:

- $F := dA + A \wedge A$ and $F^{cd} := F^c{}_{d'} \mathrm{h}^{dd'}$.
- $\mathbf{d}_{\mathrm{A}} e^a = \mathrm{d}e^a + \mathrm{A}^a{}_b \wedge e^b$ is the exterior covariant derivative.

Dynamical fields

Geometrical framework: a rank 4 vector bundle $\mathcal{V} \to \mathcal{X}$, equipped with a pseudo-metric h.

Lorentz connection

- Then A is a connection of $\mathcal{V}$ which respects the pseudo-metric h
- $A \in \Omega^1(\mathcal{X}, \mathfrak{g})$, where $\mathfrak{g} := \mathbf{so}(1,3)$

$$A = A^i_\mu \mathrm{d}x^\mu \otimes \mathfrak{l}_i := A^a_{\mu\, b} \mathrm{d}x^\mu \otimes u_a{}^b$$

Solder form

- e is a *solder form*, i.e. a rank 4 section of the vector bundle over $\mathcal{X}$ whose fiber over $\mathrm{x} \in \mathcal{X}$ is

$$\mathcal{V}_\mathrm{x} := \{ \text{ set of linear maps } T_\mathrm{x}\mathcal{X} \mapsto \mathcal{V}_\mathrm{x} \ \}$$

$$e = e^a_\mu \mathrm{d}x^\mu \otimes E_a$$

where E_a is a basis of $\mathcal{V}_\mathrm{x}$ (which is identified with the Minkowski vector space $\vec{\mathbb{M}}$)

Gauge transformations, Cartan connection

WEC action functional is invariant by gauge transformations of the form

$$(e, A) \longmapsto (g^{-1}e, g^{-1}dg + g^{-1}Ag),$$

where $g : \mathcal{X} \longrightarrow \mathfrak{G}$.

In indices:

$$\begin{aligned} e^a &\longmapsto (g^{-1})^a{}_{a'} e^{a'}, \\ A^a{}_b &\longmapsto (g^{-1})^a{}_{a'} dg^{a'}{}_b + (g^{-1})^a{}_{a'} A^{a'}{}_{b'} g^{b'}{}_b \end{aligned}$$

Lift of the variational problem

Lift on the *total space* $\mathcal{P}$ of the principal bundle of orthonormal frames on $\mathcal{V}$ (with a right action of $\mathfrak{G}$, $\mathcal{P} \times \mathfrak{G} \ni (z, g) \longmapsto z \cdot g \in \mathcal{P}$).

The lift $(\overset{0}{\boldsymbol{\eta}}, \overset{1}{\boldsymbol{\eta}}) := \pi^*(A, e)$ is a $\mathfrak{p}$-valued 1-form on the total space, i.e.

$$\boldsymbol{\eta} := (\overset{0}{\boldsymbol{\eta}}, \overset{1}{\boldsymbol{\eta}}) \in \Omega^1(\mathcal{P}, \mathfrak{p}),$$

where

- $\overset{0}{\boldsymbol{\eta}} \in \Omega^1(\mathcal{P}, \mathfrak{g})$, $\mathfrak{g}$ is the $\mathbf{so}(1,3)$ Lie algebra.
- $\overset{1}{\boldsymbol{\eta}} \in \Omega^1(\mathcal{P}, \mathfrak{t})$, where $\mathfrak{t}$, is the trivial Lie algebra of $\mathfrak{T}$, the Abelian Lie group of translations on the Minkowski spacetime.

Normalization and equivariance hypotheses

$$\begin{aligned} \rho_i \lrcorner \overset{\circ}{\boldsymbol{\eta}} &= \mathfrak{l}_i, \\ \mathcal{L}_{\rho_i}\overset{\circ}{\boldsymbol{\eta}} + [\mathfrak{l}_i, \overset{\circ}{\boldsymbol{\eta}}] &= 0, \end{aligned}$$

$$\begin{aligned} \rho_i \lrcorner \overset{1}{\boldsymbol{\eta}} &= 0, \\ \mathcal{L}_{\rho_i}\overset{1}{\boldsymbol{\eta}} + \mathfrak{l}_i \cdot \overset{1}{\boldsymbol{\eta}} &= 0. \end{aligned}$$

- For any pair $(\overset{\circ}{\boldsymbol{\eta}}, \overset{1}{\boldsymbol{\eta}})$ on $\mathcal{P}$ which satisfies the normalization and equivariance properties, for any local section $\sigma : \mathcal{X} \longrightarrow \mathcal{P}$, we obtain a pair (A, e) on $\mathcal{X}$ simply by setting $A = \sigma^* \overset{\circ}{\boldsymbol{\eta}}$ and $e = \sigma^* \overset{1}{\boldsymbol{\eta}}$.

- Conversely, given a pair (A, e) on $\mathcal{X}$ and a local section $\sigma : \mathcal{X} \longrightarrow \mathcal{P}$, this provides us with a local trivialization

$$\begin{array}{cccc} \tau : & \mathcal{P} & \longrightarrow & \mathcal{X} \times \mathfrak{G} \\ & \mathrm{z} & \longmapsto & (\mathrm{x}, g) \end{array} \quad \text{where } (\mathrm{x}, g) \text{ is s.t. } \mathrm{z} = \sigma(\mathrm{x}) \cdot g$$

We can then associate to (A, e) a pair $(\overset{\circ}{\boldsymbol{\eta}}, \overset{1}{\boldsymbol{\eta}})$ on $\mathcal{P}$ which satisfies the normalization and equivariance hypotheses are given by $\overset{\circ}{\boldsymbol{\eta}} = \tau^*(g^{-1}Ag + g^{-1}dg)$ and $\overset{1}{\boldsymbol{\eta}} = \tau^*(g^{-1}e)$.

3 10-plectic formulation of vierbein gravity

- HVDW system of equations
- De Donder-Weyl multimomentum bundle
- 10-plectic formulation of vierbein gravity
- Poincaré-Cartan 10-form

HVDW system of equations

HVDW equations

- Volterra (1890), Weyl (34), De Donder (35)
- Lepage (36,41), Dedecker (53)
- Tulczyjew (65), Kijowski, Sczcyrba (70'). Goldschmidt, Sternberg (73), García (68) García, Pérez-Rendún (73)

Multisymplectic manifold: smooth manifold $\mathcal{N}$ endowed with a *multisymplectic* $(m+1)$-form ω.

- ω is closed
- ω non degenerate,

m refers to the number of independent variables.

Hamiltonian function $\boldsymbol{H} : \mathcal{N} \longrightarrow \mathbb{R}$.

HVDW equations

Solutions of the HVDW equations by oriented m-dimensional submanifolds Γ of $\mathcal{N}$ which satisfy the condition that, $\forall \mathrm{M} \in \mathcal{N}$, $\exists (X_1, \cdots, X_m)$ of $T_{\mathrm{M}}\Gamma$ such that

$$X_1 \wedge \cdots \wedge X_m \lrcorner\, \omega = (-1)^m d\boldsymbol{H}.$$

Equivalently one can replace ω by its restriction to the level set $\boldsymbol{H}^{-1}(0)$ and describe the solutions as the submanifolds Γ of $\boldsymbol{H}^{-1}(0)$ such that $X_1 \wedge \cdots \wedge X_m \lrcorner\, \omega = 0$ everywhere (plus some independence conditions).

De Donder-Weyl multimomentum bundle

covariant configuration space: bundle $\mathcal{Z} \xrightarrow{\pi} \mathcal{X}$.

Field $:= \{\varphi \in C^\infty(\mathcal{X}, \mathcal{Z}) / \pi \circ \varphi = \mathrm{Id}\}$

- **Multimomentum phase space** Space of n-form, $\Lambda^n T^\star \mathcal{Z}$
- **Poincaré-Cartan canonical** n-form θ on $\Lambda^n T^\star \mathcal{Z}$, $\forall z \in \mathcal{Z} \quad \forall p \in \Lambda^n T^\star_z \mathcal{Z}$

$$\theta_{(z,p)}(X_1, \cdots, X_n) := p(\mathrm{d}\pi_{\mathcal{X}}(X_1), \cdots, \mathrm{d}\pi_{\mathcal{X}}(X_n))$$

- Multisymplectic $(n+1)$-form $\omega = \mathrm{d}\theta$.

DW Multimomentum phase space

- is denoted: $\boldsymbol{\mathcal{M}}_{\mathrm{DW}} := \Lambda^n_1 T^\star \mathcal{Z}$

$$\Lambda^n_1 T^\star \mathcal{Z} = \{(z, p) \in \Lambda^n T^\star \mathcal{Z} \ / \ \forall v, w \in V_z\mathcal{Z} \ v \wedge w \lrcorner\, p = 0\}.$$

- $\theta^{\mathrm{DW}} := \theta|_{\boldsymbol{\mathcal{M}}_{\mathrm{DW}}} = \varsigma\beta + p_i^\mu \mathrm{d}z^i \wedge \beta_\mu$. where $\beta = dx^1 \wedge \cdots \wedge dx^n$, $\beta_\mu = \partial_\mu \lrcorner\, \beta$.
- $\omega^{\mathrm{DW}} := \mathrm{d}\varsigma \wedge \beta + \mathrm{d}p_i^\mu \wedge \mathrm{d}z^i \wedge \beta_\mu$.
- From a variational Lagrangian problem, $\boldsymbol{H} = \varsigma + H(x^\mu, z^i, p_i^\mu)$, so that

$$\omega^{\mathrm{DW}} = -\mathrm{d}\boldsymbol{H} \wedge \beta + \mathrm{d}p_i^\mu \wedge \mathrm{d}z^i \wedge \beta_\mu$$

10-plectic formulation of vierbein gravity

Covariant configuration bundle $\mathfrak{p} \otimes T^{\star}\mathcal{P}$ over $\mathcal{P}$

Gauge field, a $\mathfrak{p}$-valued 1-form on the total space of the Lorentz bundle $\mathcal{P}$.

We denote

- Canonical 1-form $\eta \in \mathfrak{p} \otimes T_z^{*}\mathcal{P}$
- Section $\boldsymbol{\eta} \in \Gamma(\mathcal{P}, \mathfrak{p} \otimes T^{*}\mathcal{P})$

$$\Lambda_1^m T^*(\mathcal{Z}) := \Lambda_1^m T^*(\mathfrak{p} \otimes T^*\mathcal{P})$$

$$\downarrow \Pi$$

$$\mathcal{P} \xleftarrow{\pi_{\mathcal{P}}} \mathcal{Z} := \mathfrak{p} \otimes T^*\mathcal{P}$$

10-plectic formulation of vierbein gravity

Spiting $\eta = \overset{0}{\eta} + \overset{1}{\eta}$

Lie algebra $\mathfrak{p}$ and $\mathfrak{p}^*$

- the basis $(\mathfrak{l}_A)_{0 \leq A \leq 9}$ of $\mathfrak{p}$;
- the dual basis $(\mathfrak{l}^A)_{0 \leq A \leq 9}$ of $\mathfrak{p}^*$

We split $\eta = \overset{0}{\eta} + \overset{1}{\eta}$, according to the decomposition $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{t}$. Note that $\overset{1}{\eta} = \eta^a \mathfrak{l}_a = \eta^a t_a$, where $0 \leq a \leq 3$, and $\overset{0}{\eta} = \eta^i \mathfrak{l}_i = \eta^i u_i$, where $4 \leq i \leq 9$. We also set $\overset{0}{\eta}{}^a{}_b = u^a_{ib} \overset{0}{\eta}{}^i$.

The 1-form $\eta \in \mathfrak{p} \otimes T^*_z\mathcal{P}$ is written as:

$$\eta = \eta^A_\mu \mathfrak{l}_A \otimes dx^\mu + \eta^A_j \mathfrak{l}_A \otimes \gamma^j \in \mathfrak{p} \otimes T^*_z\mathcal{P}$$

The 1-form $\eta := \overset{0}{\eta} + \overset{1}{\eta} \in \mathfrak{p} \otimes T^*_z\mathcal{P}$ as

$$\eta = u^d_c \otimes (\overset{0}{\eta}{}^c_{d\mu} dx^\mu + \overset{0}{\eta}{}^c_{dj} \gamma^j) + t_c \otimes (\overset{1}{\eta}{}^c{}_\mu dx^\mu + \overset{1}{\eta}{}^c{}_j \gamma^j)$$

$(\overset{0}{\eta}{}^c_{d\mu}, \overset{0}{\eta}{}^c_{dj}, \overset{1}{\eta}{}^c{}_\mu, \overset{1}{\eta}{}^c{}_j)$ for the components of η in the basis $(u^d_c \otimes dx^\mu, u^d_c \otimes \gamma^j, t_c \otimes dx^\mu, t_c \otimes \gamma^j)$

DW 10-plectic multimomentum bundle

DW Multisymplectic manifold: $\Lambda^m_1 T^\star(\mathfrak{p} \otimes T^\star \mathcal{P})$

Coordinate functions on $\Lambda^m_1 T^*(\mathfrak{p} \otimes T^* \mathcal{P})$ are:

$$(x^\mu, g, \eta^A_\mu, \eta^A_j, \varsigma, p^{\mu\nu}_A, p^{j\nu}_A, p^{\mu j}_A, p^{jk}_A)$$

More precisely:

- (x^μ, g) for $z \in \mathcal{P}$, where $g \in \mathfrak{G}$.
- (η^A_μ, η^A_j) components of $\eta \in \mathfrak{p} \otimes T^*_z \mathcal{P}$ in the basis $(\mathfrak{l}_A \otimes \beta^\mu, \mathfrak{l}_A \otimes \gamma^j)$
- $(\varsigma, p^{\mu\nu}_A, p^{j\nu}_A, p^{\mu j}_A, p^{jk}_A)$ components of $p \in \Lambda^m_1 T^*_{(z,\eta)}(\mathfrak{p} \otimes T^* \mathcal{P})$ in the basis $(\beta \wedge \gamma, \mathrm{d}\eta^A_\mu \wedge \beta_\nu \wedge \gamma, \mathrm{d}\eta^A_j \wedge \beta_\nu \wedge \gamma, \mathrm{d}\eta^A_\mu \wedge \beta \wedge \gamma_j, \mathrm{d}\eta^A_j \wedge \beta \wedge \gamma_k)$.

Coordinate functions on $\Lambda^m_1 T^*(\mathfrak{p} \otimes T^* \mathcal{P})$ are:

$$(x^\mu, g, \overset{0}{\eta}{}^c_{d\mu}, \overset{0}{\eta}{}^c_{dj}, \overset{1}{\eta}{}^c{}_\mu, \overset{1}{\eta}{}^c{}_j, \varsigma, \overset{0}{p}{}^{d\mu\nu}_c, \overset{0}{p}{}^{dj\nu}_c, \overset{0}{p}{}^{d\mu j}_c, \overset{0}{p}{}^{djk}_c, \overset{1}{p}{}_c{}^{\mu\nu}, \overset{1}{p}{}_c{}^{j\nu}, \overset{1}{p}{}_c{}^{\mu j}, \overset{1}{p}{}_c{}^{jk})$$

where $\varsigma, \overset{0}{p}{}^{d\mu\nu}_c, \overset{0}{p}{}^{dj\nu}_c, \overset{0}{p}{}^{d\mu j}_c, \overset{0}{p}{}^{djk}_c, \overset{1}{p}{}_c{}^{\mu\nu}, \overset{1}{p}{}_c{}^{j\nu}, \overset{1}{p}{}_c{}^{\mu j}, \overset{1}{p}{}_c{}^{jk}$ components of $p \in \Lambda^m_1 T^*_{(z,\overset{0}{\eta},\overset{1}{\eta})}(\mathfrak{p} \otimes T^* \mathcal{P})$ in the basis $(\beta \wedge \gamma, \mathrm{d}\overset{0}{\eta}{}^c_{d\mu} \wedge \beta_\nu \wedge \gamma, \mathrm{d}\overset{0}{\eta}{}^c_{dj} \wedge \beta_\nu \wedge \gamma, \mathrm{d}\overset{0}{\eta}{}^c_{d\mu} \wedge \beta \wedge \gamma_j, \mathrm{d}\overset{0}{\eta}{}^c_{dj} \wedge \beta \wedge \gamma_k, \mathrm{d}\overset{1}{\eta}{}^c{}_\mu \wedge \beta_\nu \wedge \gamma, \mathrm{d}\overset{1}{\eta}{}^c{}_j \wedge \beta_\nu \wedge \gamma, \mathrm{d}\overset{1}{\eta}{}^c{}_\mu \wedge \beta \wedge \gamma_j, \mathrm{d}\overset{1}{\eta}{}^c{}_j \wedge \beta \wedge \gamma_k)$.

Poincaré-Cartan 10-form

Poincaré-Cartan 10-form

$$\begin{aligned}\theta_1^{\mathcal{Z}} &= \varsigma\beta\wedge\gamma\\ &+p_A^{\mu\nu}\mathrm{d}\eta_\mu^A\wedge\beta_\nu\wedge\gamma+p_A^{j\nu}\mathrm{d}\eta_j^A\wedge\beta_\nu\wedge\gamma\\ &+p_A^{\mu j}\mathrm{d}\eta_\mu^A\wedge\beta\wedge\gamma_j+p_A^{jk}\mathrm{d}\eta_j^A\wedge\beta\wedge\gamma_k.\end{aligned}$$

$$\begin{aligned}\theta &= \varsigma\beta\wedge\gamma\\ &+\mathring{p}_c^{d\mu\nu}\mathrm{d}\mathring{\eta}^c_{d\mu}\wedge\beta_\nu\wedge\gamma+\mathring{p}_c^{dj\nu}\mathrm{d}\mathring{\eta}^c_{dj}\wedge\beta_\nu\wedge\gamma\\ &+\mathring{p}_c^{d\mu j}\mathrm{d}\mathring{\eta}^c_{d\mu}\wedge\beta\wedge\gamma_j+\mathring{p}_c^{djk}\mathrm{d}\mathring{\eta}^c_{dj}\wedge\beta\wedge\gamma_k.\\ &+\overset{1}{p}_c{}^{\mu\nu}\mathrm{d}\overset{1}{\eta}{}^c{}_\mu\wedge\beta_\nu\wedge\gamma+\overset{1}{p}_c{}^{j\nu}\mathrm{d}\overset{1}{\eta}{}^c{}_j\wedge\beta_\nu\wedge\gamma\\ &+\overset{1}{p}_c{}^{\mu j}\mathrm{d}\overset{1}{\eta}{}^c{}_\mu\wedge\beta\wedge\gamma_j+\overset{1}{p}_c{}^{jk}\mathrm{d}\overset{1}{\eta}{}^c{}_j\wedge\beta\wedge\gamma_k,\end{aligned}$$

Work on normalized spaces i.e. $\Lambda_1^{n+r}T^*(\mathfrak{p}\otimes^{\mathrm{N}}T^*\mathcal{P})$:

Poincaré-Cartan 10-form

$$\begin{aligned}\theta_1^{\mathcal{Z}} &= \varsigma\beta\wedge\gamma\\ &+p_A^{\mu\nu}\mathrm{d}\eta_\mu^A\wedge\beta_\nu\wedge\gamma+p_A^{\mu j}\mathrm{d}\eta_\mu^A\wedge\beta\wedge\gamma_j\end{aligned}$$

$$\begin{aligned}\theta &= \varsigma\beta\wedge\gamma\\ &+\mathring{p}_c^{d\mu\nu}\mathrm{d}\mathring{\eta}^c_{d\mu}\wedge\beta_\nu\wedge\gamma+\mathring{p}_c^{d\mu j}\mathrm{d}\mathring{\eta}^c_{d\mu}\wedge\beta\wedge\gamma_j\\ &+\overset{1}{p}_c{}^{\mu\nu}\mathrm{d}\overset{1}{\eta}{}^c{}_\mu\wedge\beta_\nu\wedge\gamma+\overset{1}{p}_c{}^{\mu j}\mathrm{d}\overset{1}{\eta}{}^c{}_\mu\wedge\beta\wedge\gamma_j,\end{aligned}$$

Legendre Correspondence (LC)

The WEC (or Palatini) Lagrangian

$$L_{\mathrm{EC}}[\overset{0}{\eta}, \overset{1}{\eta}] := (1/2)\varepsilon_{abc}{}^{d}\,\overset{1}{\eta}{}^{a} \wedge \overset{1}{\eta}{}^{b} \wedge \Omega^{c}{}_{d}$$

where $\Omega := d\eta + \eta \wedge \eta$
Then, denoting $\mathbf{\Xi}^{d}_{c} = (1/2)\mathbf{\Xi}^{d\mu\nu}_{c}\beta_{\mu\nu}$ we have:

$$L_{\mathrm{EC}}[\overset{0}{\eta}, \overset{1}{\eta}] := \mathbf{\Xi}^{d}_{c} \wedge \Omega^{c}{}_{d}$$

Using the m.f. (∂_μ, ρ_i) and the representation (u^a_b, t_a) for the Poincaré Lie algebra:

$$L_{\mathrm{EC}}[\overset{0}{\eta}, \overset{1}{\eta}] = -\mathbf{\Xi}^{d\mu\nu}_{c}\lambda^{c}_{d\,\mu;\nu} + (1/2)\mathbf{\Xi}^{d\mu\nu}_{c}[\overset{0}{\eta}_{\mu}, \overset{0}{\eta}_{\nu}]^{c}_{d}$$

We thus define the image of the LC:

$$\begin{aligned}\mathcal{N}^{\mathrm{EC}} \quad := \quad & \{(z,\eta,p) \in \Lambda^{n+r}_{1} T^*(\mathfrak{p} \otimes^{\mathrm{N}} T^*\mathcal{P}); \\ & \overset{0}{p}{}^{d\mu\nu}_{c} = -\mathbf{\Xi}^{d\mu\nu}_{c}, \quad \overset{1}{p}{}^{\mu\nu}_{a} = 0.\}.\end{aligned}$$

The Hamiltonian function

$$\boldsymbol{H}(z,\eta,p) = W|_{\partial W/\partial \overset{0}{\lambda}{}^{c}_{d\,\mu;\nu}, \partial W/\partial \overset{1}{\lambda}{}^{a}_{\mu;\nu} = 0}(z,\eta,\lambda,p)$$

is finally given as

$$\begin{aligned}\boldsymbol{H}(z,\eta,p) \quad = \quad & \varsigma - (1/2)\mathbf{\Xi}^{d\mu\nu}_{c}[\overset{0}{\eta}_{\mu}, \overset{0}{\eta}_{\nu}]^{c}_{d} \\ & -(\overset{0}{p}{}^{d\mu j}_{c}[\mathfrak{l}_j, \overset{0}{\eta}_{\mu}]^{c}_{d} + \overset{1}{p}_{c}{}^{\mu j}\mathfrak{l}_j \overset{1}{\eta}{}^{c}_{\mu})\end{aligned}$$

DW 10-plectic manifold

DW 10-plectic manifold $\mathcal{M}_{\mathrm{DW}} \subset \Lambda^{10} T^*(\mathfrak{p} \otimes T^*\mathcal{P})$

$$\mathcal{M}_{\mathrm{DW}} := \mathbb{R} \oplus_{\mathcal{P}} \left(\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P}\right) \oplus_{\mathcal{P}} (\mathfrak{p} \otimes T^*\mathcal{P}) .$$

- $(z^I)_{1\leq I\leq 10}$ are local coordinates on $\mathcal{P}$, $z^I := (x^\mu, g)$ for any $z \in \mathcal{P}$, let $dz^{(10)} := dz^1 \wedge \cdots \wedge dz^{10}$ and $dz^{(8)}_{IJ} := \frac{\partial}{\partial z^J} \lrcorner \frac{\partial}{\partial z^I} \lrcorner dz^{(10)}$.

Coordinates on $\mathcal{M}_{Tot}$ is $(z^I, h, \eta^A_I, p^{IJ}_A)$

- (z^I, η^A_I) on $\mathfrak{p} \otimes T^*\mathcal{P}$
- (z^I, p^{IJ}_A) on $\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P}$. Note that $p^{IJ}_A = -p^{JI}_A$ on $\mathfrak{p}^* \otimes \Lambda^8 T^*_z\mathcal{P}$ in the basis $(\mathfrak{l}^A \otimes dz^{(8)}_{IJ})_{0\leq A\leq 9;1\leq I<J\leq 10}$.
- endow the real line $\mathbb{R}$ with the coordinate h.

Poincaré–Cartan 10-form on $\mathcal{M}_{Tot}$

We now define define the Poincaré–Cartan 10-form on $\mathcal{M}_{Tot}$

$$\boxed{\theta := \varsigma\eta^{(10)} + p_A \wedge (\eta + \tfrac{1}{2}[\eta \wedge \eta])^A},$$

where $\eta^{(10)} := \eta^1 \wedge \cdots \wedge \eta^{10}$ Alternatively,

$$\boxed{\theta := \varsigma\eta^{(10)} + p_a \wedge (d\overset{1}{\eta} + \overset{0}{\eta} \wedge \overset{1}{\eta})^a + p_i \wedge (d\overset{0}{\eta} + \overset{0}{\eta} \wedge \overset{0}{\eta})^i}.$$

Canonical forms on $\mathfrak{p} \otimes T^*\mathcal{P}$ and $\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P}$

- fiber at point $z \in \mathcal{P}$ is $\mathfrak{p} \otimes T_z^*\mathcal{P}$. A point in $(z, y) \in \mathfrak{p} \otimes T_z^*\mathcal{P}$, where $z \in \mathcal{P}$ and $y \in \mathfrak{p} \otimes T_z^*\mathcal{P}$.

Vector bundle $\mathfrak{p} \otimes T^*\mathcal{P}$ over $\mathcal{P}$

Canonical $\mathfrak{p}$-valued 1-form η

$\forall (z, y) \in \mathfrak{p} \otimes T^*\mathcal{P}, \forall v \in T_{(z,y)}(\mathfrak{p} \otimes \mathcal{P})$

$$\eta_{(z,y)}(v) = y(d\pi_{(z,y)}(v))$$

$\pi = \pi_{\mathfrak{p}\otimes T^*\mathcal{P}} : \mathfrak{p} \otimes T^*\mathcal{P} \longrightarrow \mathcal{P}$ is the c.p. map.

- for any $z \in \mathcal{P}$, we define the coordinates $(\eta_I^A)_{0\leq A\leq 9;1\leq I\leq 10}$ on the space $\mathfrak{p} \otimes T^*{}_z\mathcal{P}$ in the basis $(\mathfrak{l}_A \otimes dz^I)_{0\leq A\leq 9;1\leq I\leq 10}$.

$$\eta = \mathfrak{l}_A \eta_I^A dz^I.$$

Vector bundle $\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P}$ over $\mathcal{P}$

Canonical $\mathfrak{p}^*$-valued 8-form p on $\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P}$

$\forall (z, \mathrm{M}) \in \mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P}$,
$\forall w_1, \cdots, w_8 \in T_{(z,\mathrm{M})}(\mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P})$,

$$p_{(z,\mathrm{M})}(w_1, \cdots, w_8) = \mathrm{M}(d\pi_{(z,\mathrm{M})}(w_1), \cdots, d\pi_{(z,\mathrm{M})}(w_8)),$$

where $\pi = \pi_{\mathfrak{p}^*\otimes\Lambda^8 T^*\mathcal{P}} : \mathfrak{p}^* \otimes \Lambda^8 T^*\mathcal{P} \longrightarrow \mathcal{P}$ is the canonical projection map.
In local coordinates (z^I, p_A^{IJ}):

$$p = p_A \mathfrak{l}^A = \frac{1}{2} \mathfrak{l}^A p_A^{IJ} dz_{IJ}^{(8)}.$$

WEC–HVDW equations assuming the equivariance a priori

We obtain the following system of HVDW equations:

$$\begin{aligned} dx^\mu \wedge \varepsilon_{abc}{}^d (d\overset{\circ}{\boldsymbol{\eta}} + \overset{\circ}{\boldsymbol{\eta}} \wedge \overset{\circ}{\boldsymbol{\eta}})^c_d \wedge \overset{1}{\boldsymbol{\eta}}{}^b &= (\overset{1}{\boldsymbol{p}}{}_{a;j}^{\mu j} - \overset{1}{\boldsymbol{p}}{}_b{}^{\mu j}\mathfrak{l}_{j\,a}^{\,b})\beta \\ dx^\mu \wedge \varepsilon_{abc}{}^d (d\overset{1}{\boldsymbol{\eta}}{}^a \wedge \overset{1}{\boldsymbol{\eta}}{}^b + \overset{\circ}{\boldsymbol{\eta}}{}^a_{a'} \wedge \overset{1}{\boldsymbol{\eta}}{}^{a'} \wedge \overset{1}{\boldsymbol{\eta}}{}^b) &= (\overset{\circ}{\boldsymbol{p}}{}^d_{c\,;j}{}^{\mu j} + [\mathfrak{l}_j, \overset{\circ}{\boldsymbol{p}}{}^{\mu j}]^d_c)\beta \end{aligned}$$

We introduce:

Einstein 3-form

$$\begin{aligned} \boldsymbol{G}_a &= \varepsilon_{abc}{}^d \boldsymbol{\Omega}^c_d \wedge \overset{1}{\boldsymbol{\eta}}{}^b \\ &= \varepsilon_{abc}{}^d (d\overset{\circ}{\boldsymbol{\eta}} + \overset{\circ}{\boldsymbol{\eta}} \wedge \overset{\circ}{\boldsymbol{\eta}})^c_d \wedge \overset{1}{\boldsymbol{\eta}}{}^b \end{aligned}$$

Torsional 3-form

$$\begin{aligned} \boldsymbol{H}^d_c &= \varepsilon_{abc}{}^d (\mathbf{d}_\nabla \overset{1}{\boldsymbol{\eta}}{}^a) \wedge \overset{1}{\boldsymbol{\eta}}{}^b \\ &= \varepsilon_{abc}{}^d (d\overset{1}{\boldsymbol{\eta}}{}^a \wedge \overset{1}{\boldsymbol{\eta}}{}^b + \overset{\circ}{\boldsymbol{\eta}}{}^a_{a'} \wedge \overset{1}{\boldsymbol{\eta}}{}^{a'} \wedge \overset{1}{\boldsymbol{\eta}}{}^b) \end{aligned}$$

and the ≪*gravitational charges* ≫

related to momenta of the **Lorentz connections** $\overset{\circ}{\zeta}{}^{d\,\mu}_c = \overset{\circ}{\boldsymbol{p}}{}^d_{c\,;j}{}^{\mu j} + [\mathfrak{l}_j, \overset{\circ}{\boldsymbol{p}}{}^{\mu j}]^d_c$

and related to momenta of the **solder form** $\overset{1}{\zeta}{}_a{}^\mu = \overset{1}{\boldsymbol{p}}{}_{a;j}^{\mu j} - \overset{1}{\boldsymbol{p}}{}_b{}^{\mu j}\mathfrak{l}_{j\,a}^{\,b}$

Then collecting the notations:

$$\begin{aligned} dx^{\rho} \wedge \boldsymbol{G}_a &= dx^{\rho} \wedge (\frac{1}{3!}\varepsilon^{\sigma\lambda\mu\nu}\boldsymbol{G}_{a\lambda\mu\nu}\beta_{\sigma}) \\ &= \frac{1}{3!}\varepsilon^{\rho\lambda\mu\nu}\boldsymbol{G}_{a\lambda\mu\nu}\beta \end{aligned}$$

$$\begin{aligned} dx^{\rho} \wedge \boldsymbol{H}^d_c &= dx^{\rho} \wedge (\frac{1}{3!}\varepsilon^{\sigma\lambda\mu\nu}\boldsymbol{H}^d_{c\,\lambda\mu\nu}\beta_{\sigma}) \\ &= \frac{1}{3!}\varepsilon^{\rho\lambda\mu\nu}\boldsymbol{H}^d_{c\,\lambda\mu\nu}\beta \end{aligned}$$

Finally, we obtain the HVDW system of equation in nice and compact form:

$$\begin{aligned} \frac{1}{3!}\varepsilon^{\sigma\lambda\mu\nu}\boldsymbol{H}^b_{a\lambda\mu\nu} &= \overset{0}{\zeta}{}^{b\sigma}_{a} \\ \frac{1}{3!}\varepsilon^{\sigma\lambda\mu\nu}\boldsymbol{G}_{a\lambda\mu\nu} &= \overset{1}{\zeta}{}^{\sigma}_{a} \end{aligned}$$

WEC–HVDW equations without assuming the equivariance a priori

$$\begin{aligned} \frac{1}{3!}\varepsilon^{\sigma\lambda\mu\nu}\boldsymbol{H}^{b}_{a\lambda\mu\nu} &= \overset{0}{\zeta}{}^{b\sigma}_{a} \\ \frac{1}{3!}\varepsilon^{\sigma\lambda\mu\nu}\boldsymbol{G}_{a\lambda\mu\nu} &= \overset{1}{\zeta}{}^{\sigma}_{a} \end{aligned} \quad \Bigg\} \quad \boxed{\text{Einstein and torsional HVDW}}$$

$$\begin{aligned} (\overset{0}{\boldsymbol{\eta}}_{\mu;k} + [\mathfrak{l}_k, \overset{0}{\boldsymbol{\eta}}_{\mu}])^{c}_{d} &= 0 \\ (\overset{0}{\boldsymbol{\eta}}_{\mu;k} + [\mathfrak{l}_k, \overset{0}{\boldsymbol{\eta}}_{\mu}])^{c} &= 0 \end{aligned} \quad \Bigg\} \quad \boxed{\text{Equivariance condition for } (\overset{0}{\boldsymbol{\eta}}, \overset{1}{\boldsymbol{\eta}})}$$

We obtain **from dynamics** the local equivariance property of a Cartan connection!

Droping the normalization condition

The HVDW system of equations yields:

$$\left.\begin{aligned} \varepsilon^{\sigma\lambda\mu\nu}\boldsymbol{H}^{b}_{a\lambda\mu\nu} &= 3!\overset{0}{\zeta}{}^{b\sigma}_{a} \\ \varepsilon^{\sigma\lambda\mu\nu}\boldsymbol{G}_{a\lambda\mu\nu} &= 3!\overset{1}{\zeta}{}^{\sigma}_{a} \end{aligned}\right\}$$

Einstein and torsional HVDW

$$\left.\begin{aligned} \overset{0}{\boldsymbol{p}}{}^{d\,\mu j}_{c\,;\mu} - \overset{0}{\boldsymbol{p}}{}^{d\,kj}_{c\,;k} &= \left([\overset{0}{\boldsymbol{p}}{}^{\mu j}, \overset{0}{\boldsymbol{\eta}}_{\mu}]^{d}_{c} + \overset{1}{\boldsymbol{\eta}}_{\mu}{}^{d}\overset{1}{\boldsymbol{p}}_{c}{}^{\mu j}\right) \\ &\quad -\left([\overset{0}{\boldsymbol{p}}{}^{jk}, \overset{0}{\boldsymbol{\eta}}_{k}]^{d}_{c} + \overset{1}{\boldsymbol{\eta}}_{k}{}^{d}\overset{1}{\boldsymbol{p}}_{c}{}^{jk}\right) \\ \overset{1}{\boldsymbol{p}}{}^{\mu j}_{a;\mu} - \overset{1}{\boldsymbol{p}}{}^{kj}_{a;k} &= (\overset{1}{\boldsymbol{p}}{}^{\mu j}\overset{0}{\boldsymbol{\eta}}_{\mu})_{a} - (\overset{1}{\boldsymbol{p}}{}^{jk}\overset{0}{\boldsymbol{\eta}}_{k})_{a} \end{aligned}\right\}$$

Additional gravitational charges

$$\left.\begin{aligned} 0 &= (\overset{0}{\boldsymbol{\eta}}_{\mu;k} + [\mathfrak{l}_{k}, \overset{0}{\boldsymbol{\eta}}_{\mu}])^{c}_{d} \\ 0 &= (\overset{0}{\boldsymbol{\eta}}_{\mu;k} + [\mathfrak{l}_{k}, \overset{0}{\boldsymbol{\eta}}_{\mu}])^{c} \end{aligned}\right\}$$

Equivariance condition for $(\overset{0}{\boldsymbol{\eta}}, \overset{1}{\boldsymbol{\eta}})$

$$\left.\begin{aligned} 0 &= (\boldsymbol{\eta}_{j;\mu} - \boldsymbol{\eta}_{\mu;j} + [\boldsymbol{\eta}_{\mu}, \boldsymbol{\eta}_{j}])^{c}_{d} \\ 0 &= (\boldsymbol{\eta}_{j;\mu} - \boldsymbol{\eta}_{\mu;j} + [\boldsymbol{\eta}_{\mu}, \boldsymbol{\eta}_{j}])^{c} \\ 0 &= (\boldsymbol{\eta}_{j;k} - \boldsymbol{\eta}_{k;j} + [\boldsymbol{\eta}_{k}, \boldsymbol{\eta}_{j}] - \boldsymbol{\eta}_{n}c^{n}_{kj})^{c}_{d} \\ 0 &= (\boldsymbol{\eta}_{j;k} - \boldsymbol{\eta}_{k;j} + [\boldsymbol{\eta}_{k}, \boldsymbol{\eta}_{j}] - \boldsymbol{\eta}_{n}c^{n}_{kj})^{c} \end{aligned}\right\}$$

Normalization condition for $(\overset{0}{\boldsymbol{\eta}}, \overset{1}{\boldsymbol{\eta}})$

Outlook and perspectives

- Crystallization of space-time: recover dynamically the axiom of the fibration.
- Higher Gauge theory
- Cosmology, Dark energy.

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